CLOSED-FORM DESIGN OF MAXFLAT R-REGULAR IIR MTH-BAND FILTERS USING BLOCKWISE WAVEFORM MOMENTS

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ABSTRACT

Mth-band filters are an important class of digital filters and are often used in multirate digital signal processing systems, filter banks and wavelets. This paper considers the design problem of the maxflat R-regular IIR Mth-band filters with arbitrarily specified group delays, and gives a new closed-form expression for its filter coefficients. The filter coefficients are directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition of the maxflat R-regular Mth-band filters via the blockwise waveform moments. Finally, some examples are designed to demonstrate the effectiveness of the proposed maxflat R-regular IIR Mth-band filters.

Index Terms— Mth-band filter, maxflat response, closed-form design, IIR filter, waveform moment

1. INTRODUCTION

Mth-band filters are an important class of digital filters, and have been found numerous applications in multirate digital signal processing systems, filter banks and wavelets, and so on [1], [2], [4]. Its impulse response is required to be exactly zero-crossing except for one point. Conventionally, FIR Mth-band filters have been studied exhaustively in [1], [2], [4], [5], [6], [8], [9]. Among these methods, FIR Mth-band filters with exact linear phase have been designed in [1], [2], [4], [8], while those with arbitrarily specified group delays have also been considered in [5], [6] and [9]. It is known from the viewpoint of wavelet transform in [4] that the Mth-band filters are required to satisfy the regularity condition of wavelets. Therefore, the closed-form solutions for the maxflat R-regular FIR Mth-band filters have been presented in [4], [6], [8] and [9]. However, the design of the maxflat R-regular IIR Mth-band filters is still open.

In this paper, we consider the design problem of the maxflat R-regular IIR Mth-band filters with arbitrarily specified group delays, and give a new closed-form expression for its filter coefficients. We derive a linear system of Vandermonde equations from the regularity condition of the maxflat R-regular Mth-band filters via the blockwise waveform moments defined in [3], [8] and [9], and then obtain a set of filter coefficients by directly solving the linear system of Vandermonde equations. The proposed maxflat R-regular IIR Mth-band filters have an arbitrarily specified integer group delay response. Finally, some design examples are shown to demonstrate the effectiveness of the proposed maxflat R-regular IIR Mth-band filters.

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2. IIR MTH-BAND FILTERS

Let h_n $(0 \leq n < \infty)$ be an impulse response of IIR digital filter $H(z)$. If $H(z)$ is a Mth-band filter, its impulse response is required to be exactly zero-crossing except for one point K, i.e.,

$$h_{K+mM} = \begin{cases} \dfrac{1}{M} & (m = 0) \\ 0 & (m = \pm 1, \pm 2, \cdots) \end{cases}, \tag{1}$$

where K and M are integers, and K corresponds to the desired group delay in the passband.

Mth-band filter is required to be lowpass, and the desired frequency response is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-jK\omega} & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \tag{2}$$

Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$, i.e., $\hat{h}_n = h_{n+K}$ $(-K \leq n < \infty)$. The desired frequency response of $\hat{H}(z)$ becomes

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1 & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \tag{3}$$

By using the polyphase representation, we have

$$H(z)=\sum_{i=0}^{M-1} z^{-i}H_i(z^M), \tag{4}$$

where the ith polyphase compnent is composed by an IIR subfilter, i.e.,

$$H_i(z)=\sum_{n=0}^{N_N^i} a_n^i z^{-n} \Big/ \sum_{n=0}^{N_D^i} b_n^i z^{-n}, \tag{5}$$

where N_N^i, N_D^i are the degree of the numerator and denominator of the ith IIR subfilter, respectively, a_n^i, b_n^i are real filter coefficients, and $b_0^i = 1$. Assume that $K = L_1M + L_2$, where L_1 and L_2 are integers, and $0 \le L_2 < M$, it can be seen from the time-domain condition in (1) that $H_{L_2}(z) = z^{-L_1}/M$. Therefore, we have

$$\hat{H}(z)=z^K H(z)=\frac{1}{M}+\sum_{\substack{i=0\\ \neq L_2}}^{M-1} z^{K-i}H_i(z^M). \tag{6}$$

It can be seen from (6) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{k=0}^{M-1} \hat{H}(e^{j(\omega+\frac{2k\pi}{M})}) \equiv 1, \tag{7}$$

which means that the sum of the frequency responses at the frequency points $\omega_k = \omega + 2k\pi/M$ for $k = 0, 1, \cdots, M-1$ keep constant, regardless of what the filter coefficients a_n^i and b_n^i are. From (7), we get

$$\hat{H}(e^{j\omega_0}) = 1-\sum_{k=1}^{M-1}\hat{H}(e^{j\omega_k}). \tag{8}$$

It is clear that the frequency response at ω_0 is dependent on the frequency responses at ω_k $(k = 1, 2, \cdots, M-1)$. If its stopband response is 0, then the frequency response of $\hat{H}(z)$ will become 1 in the passband, i.e., the magnitude response of $H(z)$ is 1 and its group delay is K in the passband. Therefore, the design problem of IIR Mth-band filters with an arbitrarily specified group delay K can be reduced to the minimization of the stopband error of the frequency response of $\hat{H}(z)$.

3. MAXFLAT R-REGULAR IIR MTH-BAND FILTERS

It is known in [9] that the blockwise waveform moment around K for h_n is defined by

$$m_r(i)=\sum_{m=0}^{\infty}(mM+i-K)^r h_{mM+i}, \tag{9}$$

where $0 \le i \le M-1$. It follows from the definition of $m_r(i)$ that

$$\left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} = (-j)^r \sum_{i=0}^{M-1} m_r(i) e^{-j\frac{2(i-K)k\pi}{M}}, \tag{10}$$

i.e., the blockwise waveform moments describe the derivative behaviors of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$ $(0 \le k \le M-1)$. It is seen in (10) that the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at $\omega_k = 2k\pi/M$ are the M-point DFT (Discrete Fourier Transform) of the blockwise waveform moments $m_r(i)$. Thus, by the inverse transform, we have

$$m_r(i)=\frac{j^r}{M}\sum_{k=0}^{M-1}\left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} e^{j\frac{2(i-K)k\pi}{M}}. \tag{11}$$

It is clear that the blockwise waveform moments $m_r(i)$ bridge between the time and frequency domains by (9) and (11). Given the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$, the rth blockwise waveform moments $m_r(i)$ can be calculated via the IDFT in (11).

It is known in [4] that an Mth-band filter is said to be R-regular if it has

$$H(z) = (1+z^{-1}+\cdots+z^{-(M-1)})^R Q(z), \tag{12}$$

where $Q(z)$ is an IIR filter in this paper. It is equivalent to

$$\left.\frac{\partial^r H(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} = \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} = 0, \tag{13}$$

for $k = 1, 2, \cdots, M-1$ and $r = 0, 1, \cdots, R-1$. It is obtained from (8) and (13) that

$$\begin{cases} \left.\hat{H}(e^{j\omega})\right|_{\omega=0} = 1 & (r=0) \\ \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=0} = 0 & (r=1,2,\cdots,R-1) \end{cases}, \tag{14}$$

which means that the magnitude response $|\hat{H}(e^{j\omega})|$ and group delay $\hat{\tau}(\omega)$ at $\omega = 0$ satisfy

$$\begin{cases} \left.|\hat{H}(e^{j\omega})|\right|_{\omega=0} = 1 & (r=0) \\ \left.\frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial\omega^r}\right|_{\omega=0} = 0 & (r=1,2,\cdots,R-1) \end{cases}, \tag{15}$$

and

$$\left.\frac{\partial^r \hat{\tau}(\omega)}{\partial\omega^r}\right|_{\omega=0} = 0 \qquad (r=0,1,\cdots,R-2). \tag{16}$$

From the relationship between $H(z)$ and $\hat{H}(z)$ in (6), it is clear that $H(z)$ has flat magnitude and group delay responses at $\omega = 0$ simultaneously.

By using (11), the blockwise waveform moments $m_r(i)$ are obtained from the regularity condition in (13) and (14) as

$$m_r(i) = \begin{cases} \dfrac{1}{M} & (r=0) \\ 0 & (r=1,2,\cdots,R-1) \end{cases} . \quad (17)$$

It can be seen from (4), (5) and (6) that

$$z^{K-i}H_i(z^M) = \frac{\sum_{n=0}^{N_N^i} a_n^i z^{K-nM-i}}{\sum_{n=0}^{N_D^i} b_n^i z^{-nM}} = \sum_{m=0}^{\infty} h_{mM+i} z^{K-mM-i}. \quad (18)$$

Next, we define the waveform moments for the filter coefficients of the numerator and denominator in (18) by

$$\begin{cases} m_r^N(i) = \sum_{n=0}^{N_N^i} (nM+i-K)^r a_n^i \\ m_r^D(i) = \sum_{n=0}^{N_D^i} (nM)^r b_n^i \end{cases} . \quad (19)$$

Then, it can be obtained by taking rth derivatives of (18) and substituting $z=1$ that the condition in (17) is equivalent to

$$M m_r^N(i) = m_r^D(i) \qquad (r=0,1,\cdots,R-1). \quad (20)$$

From the definition of $m_r^N(i), m_r^D(i)$ in (19) and $b_0^i = 1$, we obtain

$$M\sum_{n=0}^{N_N^i}(nM+i-K)^r a_n^i - \sum_{n=1}^{N_D^i}(nM)^r b_n^i = \delta(r), \quad (21)$$

where $r=0,1,\cdots,R-1$, and

$$\delta(r) = \begin{cases} 1 & (r=0) \\ 0 & (r\neq 0) \end{cases} . \quad (22)$$

It should be noted that the coefficient matrix in (21) is the Vandermonde matrix with distinct elements. There is always a unique solution if $R = N_N^i + N_D^i + 1$. By using the Cramer's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$\begin{cases} a_n^i = \dfrac{(-1)^{n+1}}{M} \dfrac{N_D^i!}{n!(N_N^i-n)!} \dfrac{\prod_{m=0}^{N_N^i}(m+\frac{i-K}{M})}{\prod_{m=0}^{N_D^i}(m-n+\frac{K-i}{M})} \\ b_n^i = (-1)^n \dfrac{N_D^i!}{n!(N_D^i-n)!} \prod_{m=0}^{N_N^i} \dfrac{m+\frac{i-K}{M}}{m-n+\frac{i-K}{M}} \end{cases} . \quad (23)$$

Once M, K, N_N^i and N_D^i are given, a set of filter coefficients a_n^i and b_n^i can be easily calculated by using (23). It is clear that besides $N_N^i + N_D^i = R-1$, it is possible for the subfilter $H_i(z)$ to have different N_N^i and N_D^i for $0 \leq i \leq M-1$.

4. DESIGN EXAMPLE

In this section, we consider the design of the maxflat R-regular IIR Mth-band filters with $M=5$ and $R=10$. $N_N^i = 6$ and $N_D^i = 3$ are chosen for $0 \leq i \leq 4$. We first set $K = 29$ and get a set of filter coefficients by (23). The resulting magnitude response and group delay are shown in the solid line in Fig.1 and Fig.2, respectively, and its impulse response is shown in Fig.3(a). We have also designed two other filters with $K = 27, 16$. Their magnitude responses and group delays are shown also in Fig.1 and Fig.2. The impulse responses of $K=27$ and $K=16$ are shown in Fig.3(b) and Fig.3(c), respectively. It is seen from Fig.2 that the integer group delay responses at $\omega = 0$ can be arbitrarily specified. It should be noted that $K > 22$ for obtaining causal stable IIR filters.

5. CONCLUSION

In this paper, we have proposed a new closed-form solution for the maxflat R-regular IIR Mth-band filters. The filter coefficients have been directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition of the maxflat R-regular Mth-band filters via the blockwise waveform moments. The proposed maxflat R-regular IIR Mth-band filters have arbitrarily specified integer group delay responses.

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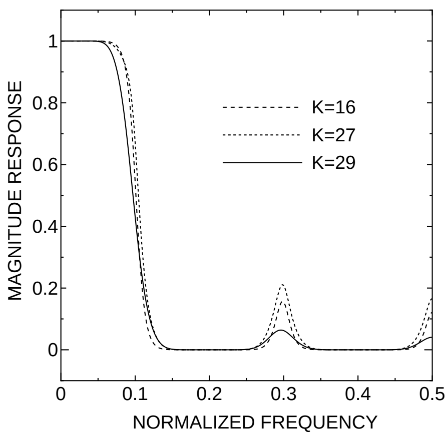

Fig. 1. Magnitude responses of the maxflat R-regular IIR Mth-band filters.

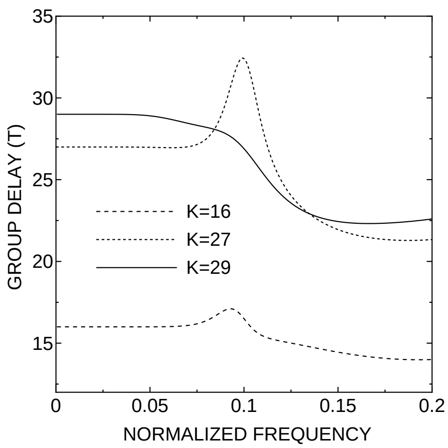

Fig. 2. Group delays of the maxflat R-regular IIR Mth-band filters.

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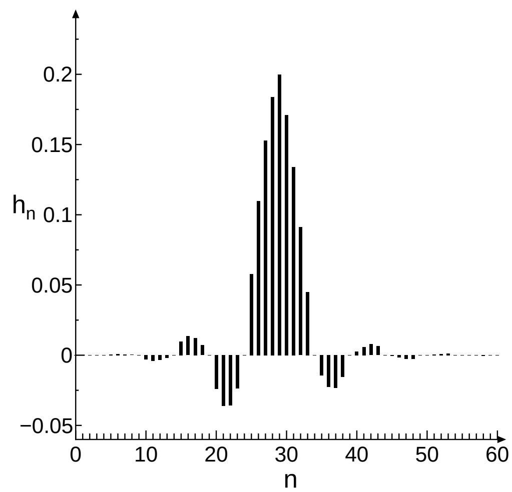
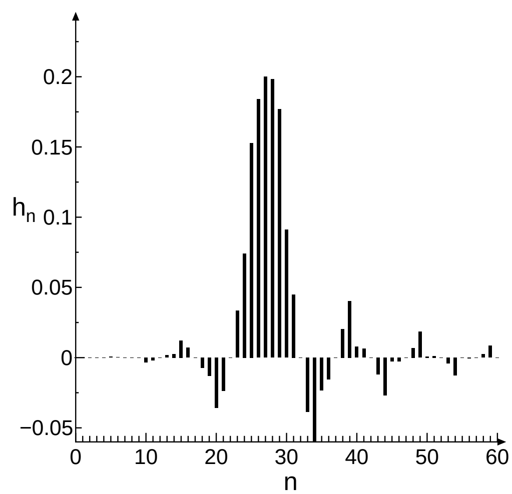
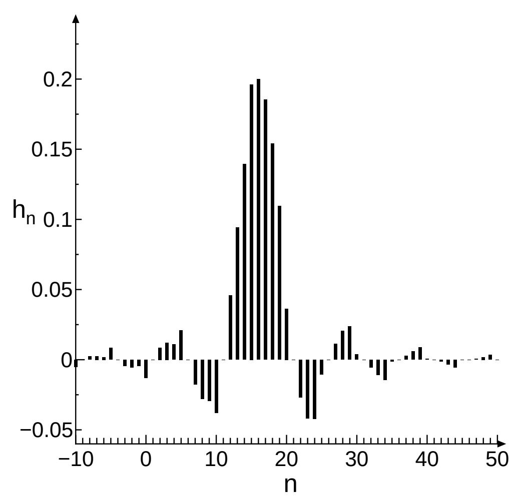


(a) $K = 29$



(b) $K = 27$



(c) $K = 16$

Fig. 3. Impulse responses of the maxflat R-regular IIR Mth-band filters. (a) $K = 29$, (b) $K = 27$, (c) $K = 16$.