Closed-Form Design of Maxflat Fractional Delay IIR Filters

Xi ZHANG

Department of Information and Communication Engineering
The University of Electro-Communications

1 Introduction

Fractional delay (FD) filters are an important class of digital filters, and are useful in various signal processing applications [1]. In this paper, the closed-form solution is proposed for the maximally flat (maxflat) FD IIR filters, which is directly derived by solving a linear system of Vandermonde equations.

2 Fractional Delay Filters

Let $H(z)$ be the transfer function of IIR filters;

$$H(z) = \sum_{n=0}^{N} a_n z^{-n} / \sum_{m=0}^{M} b_m z^{-m}, \tag{1}$$

where N and M are degree of numerator and denominator, a_n, b_m are real filter coefficients, and $b_0 = 1$.

The desired frequency response of FD filters is given by

$$H_d(e^{j\omega}) = e^{-j(K+p)\omega}, \tag{2}$$

where K is an integer delay, and p is a fractional delay in the range $[-0.5, 0.5]$. Therefore, the design problem of FD filters is the approximation of $H(e^{j\omega})$ to $H_d(e^{j\omega})$ in the specified criterion, e.g., in the minimax or maxflat sense.

3 Maxflat FD IIR Filters

To obtain a maxflat FD filter at $\omega = 0$, the frequency response of $H(z)$ must satisfy

$$\left.\frac{\partial^r H(e^{j\omega})}{\partial \omega^r}\right|_{\omega=0} = \left.\frac{\partial^r H_d(e^{j\omega})}{\partial \omega^r}\right|_{\omega=0}, \tag{3}$$

for $r = 0, 1, \cdots, N+M$. It is known that Eq.(3) is equivalent to

$$\left.\frac{\partial^r \{H(e^{j\omega})e^{j(K+p)\omega}\}}{\partial \omega^r}\right|_{\omega=0} = \left.\frac{\partial^r \{H_d(e^{j\omega})e^{j(K+p)\omega}\}}{\partial \omega^r}\right|_{\omega=0}$$

$$= \begin{cases} 1 & (r = 0) \\ 0 & (r = 1, 2, \cdots, N+M) \end{cases}. \tag{4}$$

It is obtained from Eq.(1) that

$$H(e^{j\omega})e^{j(K+p)\omega} = \frac{\sum_{n=0}^{N} a_n e^{j(K+p-n)\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}} = \frac{N(\omega)}{D(\omega)}. \tag{5}$$

Therefore, by taking rth derivatives of Eq.(5) at $\omega = 0$, it can be seen that the condition in (4) is equivalent to

$$\left.\frac{\partial^r N(\omega)}{\partial \omega^r}\right|_{\omega=0} = \left.\frac{\partial^r D(\omega)}{\partial \omega^r}\right|_{\omega=0} \quad (r = 0, 1, \cdots, N+M), \tag{6}$$

which derives a system of linear equations as follows;

$$\sum_{n=0}^{N} (K+p-n)^r a_n = \sum_{m=0}^{M} (-m)^r b_m, \tag{7}$$

for $r = 0, 1, \cdots, N+M$. It should be noted that the coefficient matrix in Eq.(7) is the Vandermonde matrix with distinct elements in the case of $p \neq 0$. Therefore, there is always a unique solution since $b_0 = 1$. By using the Cramer's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$\begin{cases} a_n = (-1)^{n+1} \dfrac{M!}{n!(N-n)!} \dfrac{\prod_{i=0}^{N} (i-K-p)}{\prod_{i=0}^{M} (i-n+K+p)} \\ b_m = (-1)^m \dfrac{M!}{m!(M-m)!} \prod_{i=0}^{N} \dfrac{i-K-p}{i-m-K-p} \end{cases}. \tag{8}$$

Once N, M, K and p are given, a set of filter coefficients a_n and b_m can be easily calculated by using Eq.(8). If we set $M = 0$ in Eq.(8), then the maxflat FD FIR filters proposed in [3] and [4] are obtained. Also if $N = M$, we have $a_n = b_{N-n}$, thus the resulting filters become the maxflat allpass filters proposed in [2]. Therefore, it is clear that the existing maxflat FD FIR and allpass filters are two special cases of the proposed maxflat FD IIR filters.

4 Conclusion

In this paper, we have proposed a new closed-form solution for the maxflat FD IIR filters. The filter coefficients have been directly derived by solving a linear system of Vandermonde equations. This new class of maxflat FD IIR filters include the existing maxflat FD FIR and allpass filters as special cases.

References

[1] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol.13, no.1, pp.30–60, Jan. 1996.

[2] J. P. Thiran, "Recursive digital filters with maximally flat group delay," *IEEE Trans. Circuits Theory*, vol.CT-18, no.6, pp.659–664, June 1971.

[3] G. S. Liu and C. H. Wei, "A new variable fractional sample delay filter with nonlinear interpolation," *IEEE Trans. Circuits and Systems II*, vol.39, no.2 pp.123–126, Feb. 1992.

[4] E. Hermanowicz, "Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers," *Electron. Lett.*, vol.28, no.20 pp.1936–1937, Sep. 1992.