Closed-Form Design of Maxflat R-Regular IIR Mth-Band Filters

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1 Introduction

Mth-band filters are an important class of digital filters and have found numerous applications in multirate signal processing systems, filter banks and wavelets [1]. This paper considers the design problem of maxflat R-regular IIR Mth-band filters, and gives the closed-form expression for its filter coefficients. The filter coefficients are directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition of the maxflat R-regular Mth-band filters via the blockwise waveform moments.

2 IIR Mth-Band Filters

Let h_n $(0 \leq n < \infty)$ be an impulse response of IIR digital filter $H(z)$. If $H(z)$ is a Mth-band filter, its impulse response is required to be exactly zero-crossing except for one point K, i.e.,

$$h_{K+mM} = \begin{cases} \dfrac{1}{M} & (m=0) \\ 0 & (m=\pm1,\pm2,\cdots) \end{cases}, \tag{1}$$

where K and M are integers, and K corresponds to the desired group delay in the passband.

Mth-band filter is required to be lowpass, and the desired frequency response is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-jK\omega} & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \tag{2}$$

Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$, i.e., $\hat{h}_n = h_{n+K}$. The desired frequency response of $\hat{H}(z)$ becomes

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1 & (\omega \in \text{passband}) \\ 0 & (\omega \in \text{stopband}) \end{cases}. \tag{3}$$

By using the polyphase representation, we have

$$H(z) = \sum_{i=0}^{M-1} z^{-i} H_i(z^M), \tag{4}$$

where

$$H_i(z) = \sum_{n=0}^{N_1^i} a_n^i z^{-n} / \sum_{n=0}^{N_2^i} b_n^i z^{-n}, \tag{5}$$

where N_1^i, N_2^i are the degree of the numerator and denominator, respectively, a_n^i, b_n^i are real filter coefficients, and $b_0^i = 1$. Assume that $K = L_1M + L_2$, where L_1, L_2 are integers, and $0 \leq L_2 \leq M-1$, it can be seen from the time-domain condition in (1) that $H_{L_2}(z) = z^{-L_1}/M$. Therefore, we have

$$\hat{H}(z) = z^K H(z) = \frac{1}{M} + \sum_{\substack{i=0 \\ \neq L_2}}^{M-1} z^{K-i} H_i(z^M). \tag{6}$$

It can be seen from (6) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{k=0}^{M-1} \hat{H}(e^{j(\omega+\frac{2k\pi}{M})}) \equiv 1, \tag{7}$$

which means that the sum of the responses at the frequency points $\omega_k = \omega + 2k\pi/M$ for $k = 0, 1, \cdots, M-1$ keep constant, regardless of what the filter coefficients are. From (7), we get

$$\hat{H}(e^{j\omega_0}) = 1 - \sum_{k=1}^{M-1} \hat{H}(e^{j\omega_k}). \tag{8}$$

It is clear that the frequency response at ω_0 is dependent on the frequency responses at ω_k $(k = 1, 2, \cdots, M-1)$. If its stopband response is 0, then the frequency response of $\hat{H}(z)$ will become 1 in the passband, i.e., the magnitude response of $H(z)$ is 1, and the group delay is K in the passband. Therefore, the design problem of IIR Mth-band filters with an arbitrarily specified K can be reduced to the minimization of the stopband error of $\hat{H}(z)$.

3 Maxflat R-regular IIR Mth-band Filters

In [5], the blockwise waveform moment around K for h_n is defined by

$$m_r(i) = \sum_{m=0}^{\infty} (mM + i - K)^r h_{mM+i}, \tag{9}$$

where $0 \leq i \leq M-1$. It follows from the definition of $m_r(i)$ that

$$\left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=\frac{2k\pi}{M}} = (-j)^r \sum_{i=0}^{M-1} m_r(i) e^{-j\frac{2(i-K)k\pi}{M}}, \tag{10}$$

i.e., the blockwise waveform moments describe the derivative behaviors of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$ $(0 \leq k \leq M-1)$. It is seen in (10) that the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at $\omega_k = 2k\pi/M$ are the M-point DFT (Discrete Fourier Transform) of the blockwise waveform moments $m_r(i)$. Thus, by the inverse transform, we have

$$m_r(i) = \frac{j^r}{M} \sum_{k=0}^{M-1} \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=\frac{2k\pi}{M}} e^{j\frac{2(i-K)k\pi}{M}}. \tag{11}$$

It is clear that the blockwise waveform moments $m_r(i)$ bridge between the time and frequency domains by (9) and (11). Given the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$, the rth blockwise waveform moments $m_r(i)$ can be calculated via the IDFT in (11).

It is known in [1] that an Mth-band filter is said to be R-regular if it has

$$H(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})^R Q(z), \qquad (12)$$

where $Q(z)$ is an IIR filter. It is equivalent to

$$\left.\frac{\partial^r H(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} = \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\frac{2k\pi}{M}} = 0, \qquad (13)$$

for $k = 1, 2, \cdots, M-1$ and $r = 0, 1, \cdots, R-1$. It is obtained from (8) and (13) that

$$\begin{cases} \left.\hat{H}(e^{j\omega})\right|_{\omega=0} & = 1 \quad (r = 0) \\ \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=0} & = 0 \quad (r = 1, 2, \cdots, R-1) \end{cases}, \qquad (14)$$

which means that the magnitude response $|\hat{H}(e^{j\omega})|$ and group delay $\hat{\tau}(\omega)$ satisfy at $\omega = 0$

$$\begin{cases} \left.|\hat{H}(e^{j\omega})|\right|_{\omega=0} & = 1 \quad (r = 0) \\ \left.\frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial\omega^r}\right|_{\omega=0} & = 0 \quad (r = 1, 2, \cdots, R-1) \end{cases}, \qquad (15)$$

and

$$\left.\frac{\partial^r \hat{\tau}(\omega)}{\partial\omega^r}\right|_{\omega=0} = 0 \qquad (r = 0, 1, \cdots, R-2). \qquad (16)$$

From the relationship between $H(z)$ and $\hat{H}(z)$ in (6), it is clear that $H(z)$ has flat magnitude and group delay responses at $\omega = 0$ simultaneously.

By using (11), the blockwise waveform moments are obtained from the regularity condition in (13) and (14) as

$$m_r(i) = \begin{cases} \frac{1}{M} & (r = 0) \\ 0 & (r = 1, 2, \cdots, R-1) \end{cases}. \qquad (17)$$

It can be obtained from (5) that

$$z^{K-i} H_i(z^M) = \frac{\sum_{n=0}^{N_1^i} a_n^i z^{K-nM-i}}{\sum_{n=0}^{N_2^i} b_n^i z^{-nM}} = \sum_{m=0}^{\infty} h_{mM+i} z^{K-mM-i}. \qquad (18)$$

Next, we define the waveform moments for the numerator and denominator in (18) by

$$\begin{cases} m_r^N(i) = \sum_{n=0}^{N_1^i} (nM + i - K)^r a_n^i \\ m_r^D(i) = \sum_{n=0}^{N_2^i} (nM)^r b_n^i \end{cases}. \qquad (19)$$

Therefore, it can be seen by taking rth derivatives of (18) and substituting $z = 1$ that the condition in (17) becomes

$$M m_r^N(i) = m_r^D(i) \qquad (r = 0, 1, \cdots, R-1). \qquad (20)$$

From the definition of $m_r^N(i), m_r^D(i)$ in (19) and $b_0^i = 1$, we obtain

$$M \sum_{n=0}^{N_1^i} (nM + i - K)^r a_n^i - \sum_{n=1}^{N_2^i} (nM)^r b_n^i = \delta(r), \qquad (21)$$

where $r = 0, 1, \cdots, R-1$, and

$$\delta(r) = \begin{cases} 1 & (r = 0) \\ 0 & (r \neq 0) \end{cases}. \qquad (22)$$

It should be noted that the coefficient matrix in (21) is the Vandermonde matrix with distinct elements. There is always a unique solution if $R = N_1^i + N_2^i + 1$. By using the Cramer's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$\begin{cases} a_n^i = \frac{(-1)^{n+1}}{M} \frac{N_2^i!}{n!(N_1^i - n)!} \frac{\prod_{m=0}^{N_1^i} (m + \frac{i-K}{M})}{\prod_{m=0}^{N_2^i} (m - n + \frac{K-i}{M})} \\ b_n^i = (-1)^n \frac{N_2^i!}{n!(N_2^i - n)!} \prod_{m=0}^{N_1^i} \frac{m + \frac{i-K}{M}}{m - n + \frac{i-K}{M}} \end{cases}. \qquad (23)$$

Once M, K, N_1^i and N_2^i are given, a set of filter coefficients a_n^i and b_n^i can be easily calculated by using (23). It is seen that besides $N_1^i + N_2^i = R-1$ must be satisfied, it is possible for $H_i(z)$ to have different N_1^i and N_2^i for $0 \leq i \leq M-1$.

4 Conclusion

In this paper, we have proposed a new closed-form solution for the maxflat R-regular IIR Mth-band filters. The filter coefficients have been directly derived by solving a linear system of Vandermonde equations, which are obtained from the regularity condition via the blockwise waveform moments.

References

[1] P. Steffen, P. N. Heller, R. A. Gopinath and C. S. Burrus, "Theory of regular M-band wavelet bases," *IEEE Trans. Signal Processing*, vol.41, no.12, pp.3497–3510, Dec. 1993.

[2] S. Pei and P. Wang, "Closed-form design of maximally flat R-regular Mth-band FIR filters," *Proc. IEEE ISCAS*, vol.1, pp.539–542, May 2000.

[3] X. Zhang and K. Amaratunga, "Closed-form design of maximally flat IIR half-band filters," *IEEE Trans. Circuits and Systems II*, vol.49, no.6, pp.409–417, June 2002.

[4] Y. Takei, K. Nagato, T. Yoshikawa and X. Zhang, "Derivative-controlled design of linear-phase FIR filters via waveform moment," *IEEE Trans. Signal Processing*, vol.51, no.10, pp.2559–2567, Oct. 2003.

[5] X. Zhang, D. Kobayashi, T. Wada, T. Yoshikawa and Y. Takei, "Closed-form design of generalized maxflat R-regular FIR Mth-band filters using waveform moments," *IEEE Trans. Signal Processing*, vol.54, no.11, pp.4214–4222, Nov. 2006.