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Design of maximally flat IIR filters with flat group delay responses

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Abstract

Digital filters with linear phase responses, that is, constant group delay responses are needed in many applications for signal and image processing. In this paper, a novel method is proposed for designing maximally flat IIR filters with flat group delay responses in the passband. First, a system of linear equations are derived from the flatness conditions of IIR filters given in the passband and stopband, respectively. Then, a set of filter coefficients can be easily obtained by simply solving this system of linear equations. In the proposed design method, the flatness of the frequency response can be specified arbitrarily. The design of lowpass filters are described in detail, and bandpass and bandstop filters are given also. Moreover, the causality and stability of the proposed maximally flat IIR filters are examined by designing various IIR filters with different group delays. It is shown from the experimental results that the obtained maximally flat IIR filters are causal stable if the group delay is set to be larger than a specific value. Finally, some design examples are presented to demonstrate the effectiveness of the proposed maximally flat IIR filters.

Keywords: IIR filter; Maximally flat filter; Flat group delay; Linear phase

1. Introduction

Digital filters with linear phase responses, that is, constant group delay responses are needed in many applications for signal processing, image processing, waveform transmission, and so on [1,2]. It is well known [1–3] that FIR filters have been used to obtain an exactly linear phase response. However, its group delay is half the filter order, thus cannot be specified arbitrarily in the design of exactly linear phase FIR filters. Moreover, a larger delay results when higher order FIR linear phase filters are needed to get a sharp magnitude response. For these reasons, the design of FIR filters with reduced delay has been also considered in [7,11]. Compared with FIR filters, IIR filters can obtain a comparable frequency response with lower filter order in general [1–3]. One of the traditional methods for designing IIR filters is to transform an analog prototype filter to the digital domain by using the bilinear transformation. However, the IIR filters obtained by the bilinear transformation have the same order numerator and denominator, and do not have a constant group delay response. Therefore, several methods have been also proposed to design IIR filters directly in the digital domain [5,6,8–10,12].

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In the design of maximally flat (MF) filters, MF IIR filters can be obtained by transforming analog Butterworth filters, while MF FIR filters were originally addressed in [4]. The generalized digital Butterworth filters with different order numerator and denominator, including the classic Butterworth IIR filters and FIR filters, have been also presented in [8], however, their group delay responses cannot be specified arbitrarily. In [5,6], a special class of IIR filters, all-pole filters have been used to approximate the passband group delay response to a constant in the maximally flat sense. In [9], such an all-pole filter with constant group delay response was chosen to be the denominator, while a mirror image polynomial was used as the numerator to have a flat magnitude response. Therefore, the degree of flatness in the magnitude and group delay responses are separately determined by the numerator and denominator. In [10], IIR lowpass filters with MF magnitude responses, whose numerator is a mirror image polynomial, have been presented by using the above-mentioned all-pole filters, where the flat group delay response can be also obtained by appropriately adjusting the delay of the filter, resulting in a parallel structure of a delay and an allpass filter. In [12], MF IIR halfband filters, a special case of IIR filters, have been derived directly from the maximal flatness conditions given in the passband and stopband, including the flatness condition imposed on the group delay response.

In this paper, we consider a more general class of IIR filters with different order numerator and denominator, whose numerator is not restricted to be a mirror image polynomial. We propose a novel method for designing MF IIR filters with flat group delay responses in the passband. In the proposed design method, a system of linear equations are derived directly from the flatness conditions of IIR filters given in the passband and stopband. Therefore, a set of filter coefficients can be easily calculated by simply solving this system of linear equations. The feature of this method is that the flatness of the frequency response can be specified arbitrarily. This design method can be applied to not only the design of lowpass filters, but also highpass, bandpass and bandstop filters. Moreover, the causality and stability of the proposed MF IIR filters are examined by designing various IIR filters with different group delays. It is shown from the experimental results that the proposed MF IIR filters become causal and stable when the desired group delay is chosen above a certain value. Finally, some examples are designed to demonstrate the effectiveness of the proposed MF IIR filters.

This paper is organized as follows. In Section 2, the transfer function and frequency response of general IIR filters are described. Section 3 proposes a design method for MF lowpass filters with flat group delay responses. The proposed method is extended to the design of MF bandpass and bandstop filters in Section 4. In Section 5, the causality and stability of the proposed MF IIR filters are examined, and some examples are shown to investigate their frequency responses. Conclusions are given in Section 6.

2. The transfer function of IIR filters

The transfer function of an IIR filter $H(z)$ is defined by

$$H(z) = \frac{A(z)}{B(z)} = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}}, \tag{1}$$

where N, M are numerator and denominator orders, respectively, a_n and b_m are real filter coefficients, and $b_0 = 1$. It is noted that the classic IIR filters obtained by the bilinear transformation have the same order numerator and denominator, i.e., $N = M$. Here we use the numerator and denominator of different order, which makes us flexible to design the filter. If $N = 0$, it is an all-pole filter in [5], while if $M = 0$, it degrades to FIR filter. In this paper, we do not restrict the numerator $A(z)$ to be a mirror image polynomial, and will consider the design of general IIR filters.

The frequency response of $H(z)$ is generally a complex-valued function of the normalized frequency ω;

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)} = \frac{\sum_{n=0}^{N} a_n e^{-jn\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}}, \tag{2}$$

where its magnitude and phase responses are given, respectively, as

$$|H(e^{j\omega})| = \sqrt{\frac{(\sum_{n=0}^{N} a_n \cos n\omega)^2 + (\sum_{n=0}^{N} a_n \sin n\omega)^2}{(\sum_{m=0}^{M} b_m \cos m\omega)^2 + (\sum_{m=0}^{M} b_m \sin m\omega)^2}}, \tag{3}$$

$$\theta(\omega) = -\tan^{-1} \frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega} + \tan^{-1} \frac{\sum_{m=0}^{M} b_m \sin m\omega}{\sum_{m=0}^{M} b_m \cos m\omega}. \tag{4}$$

Then its group delay response is obtained by

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega}. \tag{5}$$

It should be noted that unlike [9,10], the numerator will contribute to the group delay also, since it is not restricted to be a mirror image polynomial.

3. Design of MF lowpass filters

In this section, we consider the design of MF lowpass filters with flat group delay response. The desired frequency response of lowpass filters is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\tau_0\omega} & (0 \leqslant \omega \leqslant \omega_p), \\ 0 & (\omega_s \leqslant \omega \leqslant \pi), \end{cases} \tag{6}$$

where τ_0 is the desired group delay in passband, and ω_p, ω_s are the cutoff frequencies of passband and stopband, respectively. In passband, the flatness conditions of the magnitude and group delay responses are given by

$$\begin{cases} ||H(e^{j\omega})||_{\omega=0} = 1, \\ \left.\dfrac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K-1), \end{cases} \tag{7}$$

$$\begin{cases} \tau(\omega)|_{\omega=0} = \tau_0, \\ \left.\dfrac{\partial^i \tau(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K-2), \end{cases} \tag{8}$$

where K is a parameter that controls the degree of flatness in passband. In stopband, the flatness condition of the magnitude response is

$$\left.\frac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, L-1), \tag{9}$$

where L is a parameter that controls the degree of flatness in stopband.

First, we consider the flatness conditions in passband. Let $\hat{H}(e^{j\omega})$ be a noncausal shifted version of $H(e^{j\omega})$;

$$\hat{H}(e^{j\omega}) = H(e^{j\omega})e^{j\tau_0\omega} = \frac{A(e^{j\omega})e^{j\tau_0\omega}}{B(e^{j\omega})}, \tag{10}$$

which means

$$\begin{cases} |\hat{H}(e^{j\omega})| = |H(e^{j\omega})|, \\ \hat{\tau}(\omega) = \tau(\omega) - \tau_0, \end{cases} \tag{11}$$

where $|\hat{H}(e^{j\omega})|$ and $\hat{\tau}(\omega)$ are the magnitude and group delay responses of $\hat{H}(e^{j\omega})$, respectively. Therefore, the flatness conditions in Eqs. (7) and (8) become

$$\begin{cases} ||\hat{H}(e^{j\omega})||_{\omega=0} = 1, \\ \left.\dfrac{\partial^i |\hat{H}(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K-1), \end{cases} \tag{12}$$

$$\left.\frac{\partial^i \hat{\tau}(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \ldots, K-2). \tag{13}$$

Let $\hat{\theta}(\omega)$ be the phase response of $\hat{H}(e^{j\omega})$. Since the phase is 0 at $\omega = 0$ for the digital filters with real-valued coefficients, that is, $\hat{\theta}(0) = 0$, Eq. (13) becomes

$$\left.\frac{\partial^i \hat{\theta}(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \ldots, K-1). \tag{14}$$

Theorem 1. *The flatness conditions in Eqs.* (12) *and* (14) *are equivalent to*

$$\begin{cases} \hat{H}(e^{j\omega})|_{\omega=0} = 1, \\ \left.\dfrac{\partial^i \hat{H}(e^{j\omega})}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K-1). \end{cases} \tag{15}$$

Proof. Since $\hat{H}(e^{j\omega}) = |\hat{H}(e^{j\omega})|e^{j\hat{\theta}(\omega)}$, $\hat{H}(1) = 1$ means $|\hat{H}(1)| = 1$ and $\hat{\theta}(0) = 0$, and vice versa. We have

$$\begin{aligned} \frac{\partial \hat{H}(e^{j\omega})}{\partial \omega} &= \frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} e^{j\hat{\theta}(\omega)} + |\hat{H}(e^{j\omega})| \frac{\partial e^{j\hat{\theta}(\omega)}}{\partial \omega} \\ &= \left[\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} + j|\hat{H}(e^{j\omega})| \frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right] e^{j\hat{\theta}(\omega)}, \end{aligned} \tag{16}$$

then

$$\left.\frac{\partial \hat{H}(e^{j\omega})}{\partial \omega}\right|_{\omega=0} = \left.\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega}\right|_{\omega=0} + j\left.\frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right|_{\omega=0}. \tag{17}$$

Thus, $\partial \hat{H}(e^{j\omega})/\partial \omega|_{\omega=0} = 0$ is equivalent to $\partial |\hat{H}(e^{j\omega})|/\partial \omega|_{\omega=0} = 0$ and $\partial \hat{\theta}(\omega)/\partial \omega|_{\omega=0} = 0$. Similarly,

$$\left.\frac{\partial^2 \hat{H}(e^{j\omega})}{\partial \omega^2}\right|_{\omega=0} = \left.\frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2}\right|_{\omega=0} + j\left.\frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\right|_{\omega=0}, \tag{18}$$

$$\vdots$$

$$\left.\frac{\partial^i \hat{H}(e^{j\omega})}{\partial \omega^i}\right|_{\omega=0} = \left.\frac{\partial^i |\hat{H}(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} + j\left.\frac{\partial^i \hat{\theta}(\omega)}{\partial \omega^i}\right|_{\omega=0}. \tag{19}$$

It can be seen that $\partial^i \hat{H}(e^{j\omega})/\partial\omega^i|_{\omega=0} = 0$ is equivalent to $\partial^i|\hat{H}(e^{j\omega})|/\partial\omega^i|_{\omega=0} = 0$ and $\partial^i\hat{\theta}(\omega)/\partial\omega^i|_{\omega=0} = 0$. Therefore, it has been proven that Eqs. (12) and (14) are equivalent to Eq. (15). □

It is clear that Theorem 1 holds for any $\omega = \omega_p$, where ω_p is the specified frequency point in the passband.

According to Theorem 1 in [9], the condition in Eq. (15) is satisfied, if the numerator and denominator of $\hat{H}(e^{j\omega})$ satisfy

$$\left.\frac{\partial^i\{A(e^{j\omega})e^{j\tau_0\omega}\}}{\partial\omega^i}\right|_{\omega=0} = \left.\frac{\partial^i B(e^{j\omega})}{\partial\omega^i}\right|_{\omega=0} \quad (i = 0, 1, \ldots, K-1). \tag{20}$$

From Eq. (20), we then get

$$\sum_{n=0}^{N}(n-\tau_0)^i a_n - \sum_{m=0}^{M} m^i b_m = 0 \quad (i = 0, 1, \ldots, K-1). \tag{21}$$

Next, we consider the flatness condition in stopband. According to [9, Theorem 2], the condition in Eq. (9) is equivalent to

$$\left.\frac{\partial^i A(e^{j\omega})}{\partial\omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, L-1), \tag{22}$$

which means that L zeros are located at $z = -1$, i.e., $\omega = \pi$. Therefore, the degree of flatness L in stopband is at most equal to the order of numerator N, that is, $L \leqslant N$. From Eq. (22), we have

$$\sum_{n=0}^{N}(-1)^n n^i a_n = 0 \quad (i = 0, 1, \ldots, L-1). \tag{23}$$

It is clear that Eqs. (21) and (23) are a system of linear equations. If $K + L = N + M + 1$, then a set of filter coefficients can be easily obtained by solving Eqs. (21) and (23), due to $b_0 = 1$. It should be noted that K must be chosen as $K > M$, since $L \leqslant N$. Therefore, MF IIR lowpass filters with flat group delay can be easily designed. It should be noted also that if we choose $M = 0$, then MF FIR filters with flat group delay can be easily obtained.

4. Design of bandpass and bandstop filters

In the preceding section, we have described the design method of MF lowpass filters with flat group delay. MF highpass filters can be similarly designed just by changing the flatness conditions imposed on $\omega = 0$ and π. Alternatively, MF highpass filters can be readily derived from the MF lowpass filters designed in the preceding section by replacing z with $-z$ in the transfer functions. Therefore, the discussion of MF highpass filters with flat group delay is omitted here. In this section, we will consider the design of MF bandpass and bandstop filters with flat group delay response.

4.1. MF bandpass filters

The desired frequency response of bandpass filters is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j(\tau_0\omega+\theta_0)} & (\omega_{p1} \leqslant \omega \leqslant \omega_{p2}), \\ 0 & (0 \leqslant \omega \leqslant \omega_{s1}, \omega_{s2} \leqslant \omega \leqslant \pi), \end{cases} \tag{24}$$

where ω_{p1}, ω_{p2} ($\omega_{p1} < \omega_{p2}$) are the cutoff frequencies of passband, and ω_{s1}, ω_{s2} ($\omega_{s1} < \omega_{s2}$) are the cutoff frequencies of stopband, respectively. θ_0 is an initial phase.

In the passband, the flatness conditions of the magnitude and group delay responses are

$$\begin{cases} |H(e^{j\omega})||_{\omega=\omega_p} = 1, \\ \left.\dfrac{\partial^i|H(e^{j\omega})|}{\partial\omega^i}\right|_{\omega=\omega_p} = 0 \quad (i = 1, 2, \ldots, K-1) \end{cases} \tag{25}$$

and

$$\begin{cases} \tau(\omega)|_{\omega=\omega_p} = \tau_0, \\ \left.\dfrac{\partial^i\tau(\omega)}{\partial\omega^i}\right|_{\omega=\omega_p} = 0 \quad (i = 1, 2, \ldots, K-2), \end{cases} \tag{26}$$

where K is a parameter that controls the degree of flatness at the given frequency point $\omega = \omega_p$, and $\omega_{p1} \leqslant \omega_p \leqslant \omega_{p2}$.

Let $\hat{H}(e^{j\omega}) = H(e^{j\omega})e^{j(\tau_0\omega+\theta_0)}$, then the flatness conditions in Eqs. (25) and (26) become

$$\begin{cases} |\hat{H}(e^{j\omega})||_{\omega=\omega_p} = 1, \\ \left.\dfrac{\partial^i|\hat{H}(e^{j\omega})|}{\partial\omega^i}\right|_{\omega=\omega_p} = 0 \quad (i = 1, 2, \ldots, K-1), \end{cases} \tag{27}$$

$$\left.\frac{\partial^i\hat{\tau}(\omega)}{\partial\omega^i}\right|_{\omega=\omega_p} = 0 \quad (i = 0, 1, \ldots, K-2). \tag{28}$$

Since the phase of $H(z)$ is required to be equal to the desired phase at $\omega = \omega_p$, that is,

$\theta(\omega_p) = -(\tau_0\omega_p + \theta_0)$, thus, $\hat{\theta}(\omega_p) = 0$, and we have

$$\left.\frac{\partial^i \hat{\theta}(\omega)}{\partial \omega^i}\right|_{\omega=\omega_p} = 0 \quad (i = 0, 1, \ldots, K-1). \tag{29}$$

Similarly, according to Theorem 1 in this paper and Theorem 1 in [9], Eqs. (27) and (29) are equivalent to

$$\left.\frac{\partial^i \{A(e^{j\omega}) e^{j(\tau_0\omega+\theta_0)}\}}{\partial \omega^i}\right|_{\omega=\omega_p} = \left.\frac{\partial^i B(e^{j\omega})}{\partial \omega^i}\right|_{\omega=\omega_p}$$
$$(i = 0, 1, \ldots, K-1), \tag{30}$$

that is,

$$\sum_{n=0}^{N} (n-\tau_0)^i e^{j\{(\tau_0-n)\omega_p+\theta_0\}} a_n - \sum_{m=0}^{M} m^i e^{-jm\omega_p} b_m = 0, \tag{31}$$

which is separated into the real and imaginary parts;

$$\sum_{n=0}^{N} (n-\tau_0)^i \cos\{(\tau_0-n)\omega_p+\theta_0\} a_n - \sum_{m=0}^{M} m^i \cos(m\omega_p) b_m = 0, \tag{32}$$

and

$$\sum_{n=0}^{N} (n-\tau_0)^i \sin\{(\tau_0-n)\omega_p+\theta_0\} a_n + \sum_{m=0}^{M} m^i \sin(m\omega_p) b_m = 0. \tag{33}$$

In the stopband, the flatness conditions of the magnitude response are

$$\begin{cases} \left.\dfrac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 & (i = 0, 1, \ldots, L_1 - 1), \\ \left.\dfrac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=\pi} = 0 & (i = 0, 1, \ldots, L_2 - 1), \end{cases} \tag{34}$$

where L_1, L_2 are parameters that control the degree of flatness at $\omega = 0$ and π, respectively. Similarly, the flatness conditions in Eq. (34) derive a set of linear equations;

$$\begin{cases} \sum_{n=0}^{N} n^i a_n = 0 & (i = 0, 1, \ldots, L_1 - 1), \\ \sum_{n=0}^{N} (-1)^n n^i a_n = 0 & (i = 0, 1, \ldots, L_2 - 1). \end{cases} \tag{35}$$

This means that L_1 and L_2 zeros are located at $z = 1$ and -1, respectively. Thus, it is impossible that the degree of flatness $L_1 + L_2$ in the stopband is larger than the order of numerator N, that is, $L_1 + L_2 \leqslant N$. When $2K + L_1 + L_2 = N + M + 1$, Eqs. (32), (33) and (35) can be easily solved to obtain the filter coefficients, since $b_0 = 1$. Therefore, K must be chosen as $K > M/2$, and then MF bandpass filters with flat group delay can be designed directly.

4.2. MF bandstop filters

The desired frequency response of bandstop filters is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\tau_1\omega} & (0 \leqslant \omega \leqslant \omega_{p1}), \\ 0 & (\omega_{s1} \leqslant \omega \leqslant \omega_{s2}), \\ e^{-j(\tau_2\omega+\theta_0)} & (\omega_{p2} \leqslant \omega \leqslant \pi), \end{cases} \tag{36}$$

where ω_{p1}, ω_{p2} ($\omega_{p1} < \omega_{p2}$) are the cutoff frequencies of passband, and ω_{s1}, ω_{s2} ($\omega_{s1} < \omega_{s2}$) are the cutoff frequencies of stopband. τ_1, τ_2 are the desired group delays in first and second passbands, respectively, and can be set to be different. Since we consider only the digital filters with real-valued coefficients, θ_0 must be chosen to satisfy $\theta_0 + \tau_2\pi = k\pi$. It is noted that $H(-1) = 1$ if k is even, while $H(-1) = -1$ if k is odd. That is, $H(z)$ have the same phase at $\omega = 0$ and $\omega = \pi$ if k is even, while there is a phase difference of π if k is odd.

In the first passband, the flatness conditions of the magnitude and group delay responses are

$$\begin{cases} |H(e^{j\omega})||_{\omega=0} = 1, \\ \left.\dfrac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K_1 - 1), \end{cases} \tag{37}$$

$$\begin{cases} \tau(\omega)|_{\omega=0} = \tau_1, \\ \left.\dfrac{\partial^i \tau(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, K_1 - 2), \end{cases} \tag{38}$$

where K_1 is a parameter that controls the degree of flatness at $\omega = 0$. Let $\hat{H}_1(e^{j\omega}) = H(e^{j\omega}) e^{j\tau_1\omega}$, then the flatness conditions in Eqs. (37) and (38) become

$$\left.\frac{\partial^i \{A(e^{j\omega}) e^{j\tau_1\omega}\}}{\partial \omega^i}\right|_{\omega=0} = \left.\frac{\partial^i B(e^{j\omega})}{\partial \omega^i}\right|_{\omega=0}$$
$$(i = 0, 1, \ldots, K_1 - 1), \tag{39}$$

that is,

$$\sum_{n=0}^{N}(n-\tau_1)^i a_n - \sum_{m=0}^{M} m^i b_m = 0$$
$$(i = 0, 1, \ldots, K_1 - 1). \tag{40}$$

In the second passband, the flatness conditions of the magnitude and group delay responses are

$$\begin{cases} |H(\mathrm{e}^{\mathrm{j}\omega})||_{\omega=\pi} = 1, \\ \left.\dfrac{\partial^i |H(\mathrm{e}^{\mathrm{j}\omega})|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 1, 2, \ldots, K_2 - 1), \end{cases} \tag{41}$$

$$\begin{cases} \tau(\omega)|_{\omega=\pi} = \tau_2, \\ \left.\dfrac{\partial^i \tau(\omega)}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 1, 2, \ldots, K_2 - 2), \end{cases} \tag{42}$$

where K_2 is a parameter that controls the degree of flatness at $\omega = \pi$. Let $\hat{H}_2(\mathrm{e}^{\mathrm{j}\omega}) = H(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}(\tau_2\omega+\theta_0)}$, the flatness conditions in Eqs. (41) and (42) become

$$\left.\frac{\partial^i \{A(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}(\tau_2\omega+\theta_0)}\}}{\partial \omega^i}\right|_{\omega=\pi} = \left.\frac{\partial^i B(\mathrm{e}^{\mathrm{j}\omega})}{\partial \omega^i}\right|_{\omega=\pi}$$
$$(i = 0, 1, \ldots, K_2 - 1) \tag{43}$$

that is,

$$\sum_{n=0}^{N}(-1)^{n-k}(n-\tau_2)^i a_n - \sum_{m=0}^{M}(-1)^m m^i b_m = 0$$
$$(i = 0, 1, \ldots, K_2 - 1), \tag{44}$$

where k is chosen to be $(-1)^k = 1$ or $(-1)^k = -1$ for having the same phase or phase difference of π at $\omega = 0$ and π.

In the stopband, the flatness condition of the magnitude response is given by

$$\left.\frac{\partial^i |H(\mathrm{e}^{\mathrm{j}\omega})|}{\partial \omega^i}\right|_{\omega=\omega_s} = 0 \quad (i = 0, 1, \ldots, L - 1), \tag{45}$$

where L is a parameter that controls the degree of flatness at the given frequency point $\omega = \omega_s$, and $\omega_{s1} \leqslant \omega_s \leqslant \omega_{s2}$. Similarly, the flatness condition in Eq. (45) derives a set of linear equations;

$$\sum_{n=0}^{N} n^i \mathrm{e}^{-\mathrm{j}n\omega_s} a_n = 0 \quad (i = 0, 1, \ldots, L - 1) \tag{46}$$

that is,

$$\begin{cases} \sum\limits_{n=0}^{N} n^i \cos(n\omega_s) a_n = 0 \quad (i = 0, 1, \ldots, L - 1), \\ \sum\limits_{n=0}^{N} n^i \sin(n\omega_s) a_n = 0 \quad (i = 0, 1, \ldots, L - 1). \end{cases} \tag{47}$$

This means that L zeros are located at $\omega = \pm\omega_s$, thus, $L \leqslant N/2$. When $2L + K_1 + K_2 = N + M + 1$, Eqs. (40), (44) and (47) can be easily solved to obtain the filter coefficients, since $b_0 = 1$. Therefore, $K_1 + K_2$ must be chosen as $K_1 + K_2 > M$, and then MF bandstop filters with flat group delay can be designed directly.

5. Design examples

In this section, we give some design examples of the proposed MF IIR filters to investigate their frequency responses, and then examine the causality and stability by designing various IIR filters with different group delays.

Example 1. We consider the design of MF IIR lowpass filters with $N = 7, M = 3$, and the desired group delay $\tau_0 = 5.2$. The degrees of flatness in the passband and stopband are set to be $K = 5, 6, 7, 8$ and $L = 6, 5, 4, 3$, respectively. Since $K + L = N + M + 1$, the filter coefficients can be easily obtained by solving a system of linear equations in Eqs. (21)

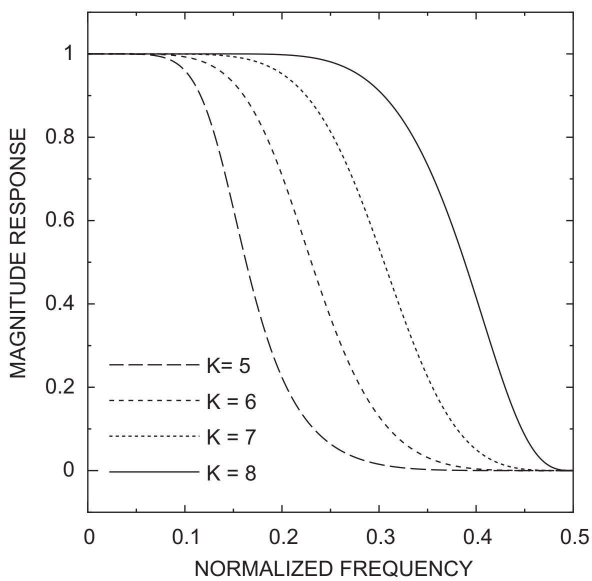

Fig. 1. Magnitude responses of MF IIR lowpass filters in Example 1.

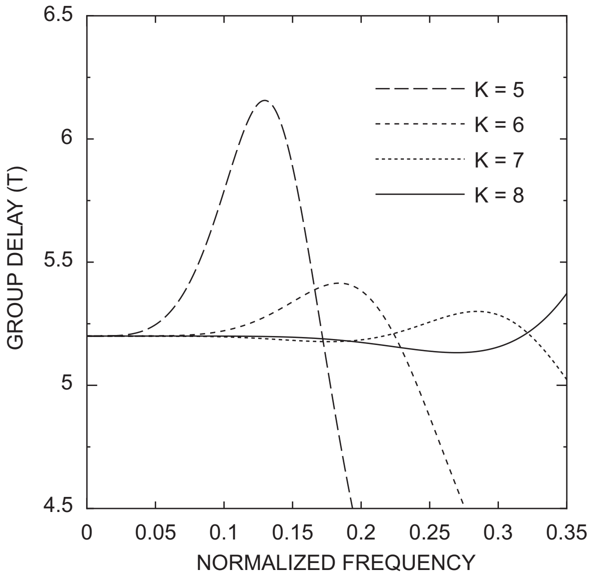

Fig. 2. Group delays of MF IIR lowpass filters in Example 1.

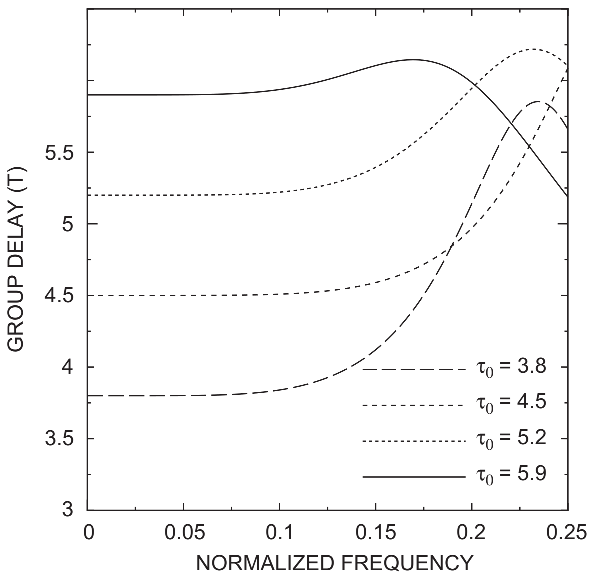

Fig. 4. Group delays of MF IIR lowpass filters in Example 2.

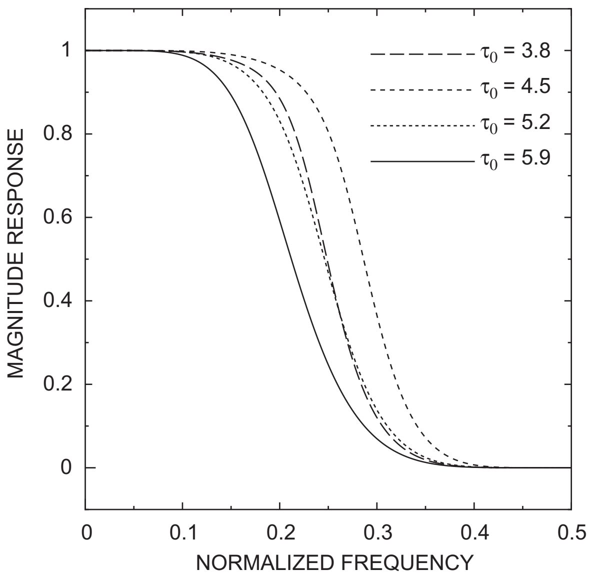

Fig. 3. Magnitude responses of MF IIR lowpass filters in Example 2.

and (23). The resulting IIR filters are found to be causal stable, and the magnitude and group delay responses are shown in Figs. 1 and 2, respectively. It is seen in Figs. 1 and 2 that the proposed MF IIR filters have more flat magnitude and group delay responses in passband with an increasing K.

Example 2. We consider the design of MF IIR lowpass filters with $N = 8, M = 3$, and the degrees of flatness $K = L = 6$ in the passband and stopband. The desired group delay is set to be $\tau_0 = 5.9, 5.2, 4.5, 3.8$, respectively. The resulting magnitude and group delay responses are shown in Figs. 3 and 4, respectively. It is seen in Fig. 3 that the magnitude response varies with the desired group delay τ_0, regardless of the degree of flatness, while the desired flat group delay responses have been obtained in Fig. 4. It is also found that the IIR filter with $\tau_0 = 3.8$ is not causal stable, having one pole outside the unit circle, whereas other three filters are causal stable. Therefore, this is thought that the desired group delay τ_0 influences the causality and stability of IIR filters [12]. Next, we have designed MF IIR lowpass filters with $N = 5, M = 2$, and $K = 3, L = 5$ to examine the causality and stability by changing the desired group delay from $\tau_0 = -10$ to $\tau_0 = 10$ with an increment $\Delta\tau = 0.1$. The trajectory of the poles with the desired group delay τ_0 is shown in Fig. 5. It is seen that when τ_0 increases, the poles move from the outside to the inside of the unit circle, crossing the unit circle at $\tau_0 = 1.5$. If $\tau_0 \to \infty$, then the poles $\to 1$. Therefore, it is clear that this IIR filter is causal and stable when $\tau_0 > 1.5$. We have investigated many MF IIR filter with different group delays, and thus found that the proposed MF IIR filters become causal stable if we choose the desired group delay τ_0 to be larger than a specific value [12], which is dependent on the design specification

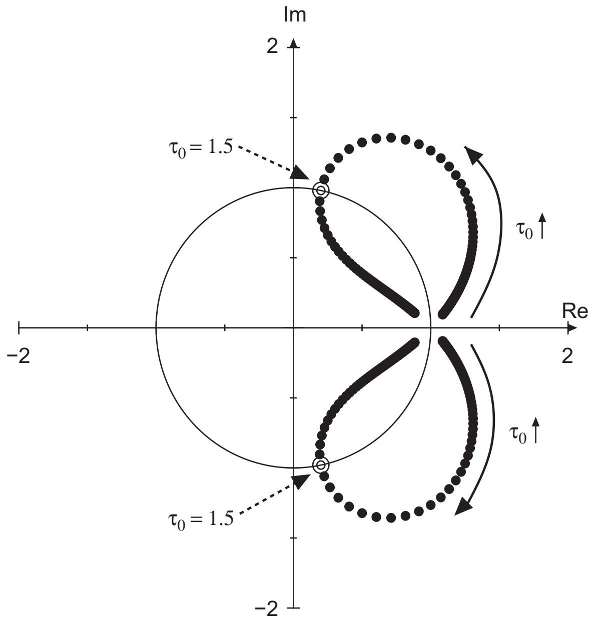

Fig. 5. Trajectory of the poles with the desired group delay.

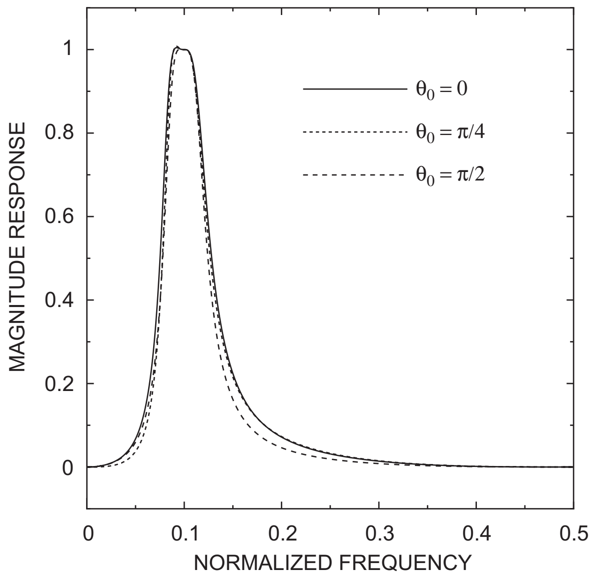

Fig. 6. Magnitude responses of MF IIR bandpass filters in Example 3.

N, M, K, L, and whose relationship needs to be further investigated.

Example 3. We consider the design of MF IIR bandpass filters with $N = 6, M = 4$, the desired group delay $\tau_0 = 12.8$, and the degrees of flatness $K = 3$ at $\omega_p = 0.2\pi$ in passband and $L_1 = 2, L_2 = 3$ in stopband. The initial phase, which is absent in the lowpass filter design, is set to be $\theta = 0, 0.25\pi, 0.5\pi$, respectively. The resulting magnitude and group delay responses are shown in Figs. 6 and 7, respectively. It is seen that the magnitude and group delay responses slightly vary with the initial phase θ_0, regardless of the degree of flatness and the desired group delay. It is found also that the obtained MF IIR bandpass filters are causal stable.

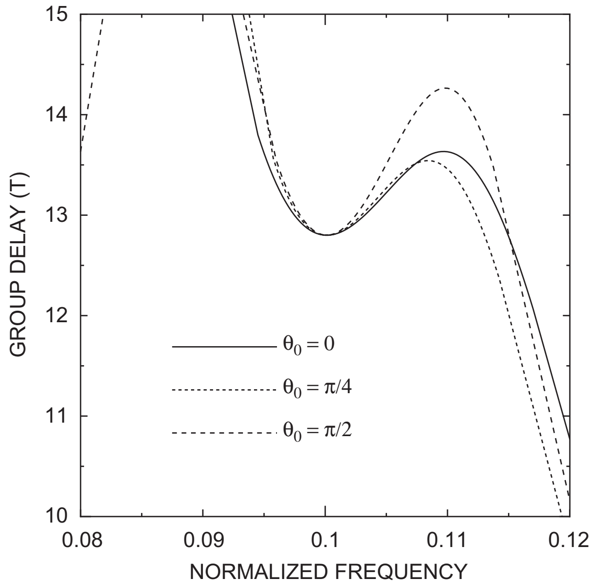

Fig. 7. Group delays of MF IIR bandpass filters in Example 3.

Example 4. We consider the design of MF IIR bandstop filters with $N = 9, M = 4$, the desired group delays $\tau_1 = 8.8$, $\tau_2 = 9.2$, and the degree of flatness $L = 3$ at $\omega_s = 0.5\pi$ in stopband. The degrees of flatness in passband are set to be $K_1 = 2, 4, 6$ and $K_2 = 6, 4, 2$, respectively. $\theta_0 = -0.2\pi$ has been chosen such that $k = 9$ is odd and then $H(-1) = -1$. The resulting magnitude and group delay responses are shown in Figs. 8 and 9, respectively, and the obtained MF IIR bandstop filters are causal stable.

6. Conclusions

In this paper, we have proposed a new method for designing a more general class of MF IIR filters with flat group delay responses in the passband, whose numerator and denominator are of different order and whose numerator is not restricted to be a mirror image polynomial. The proposed design

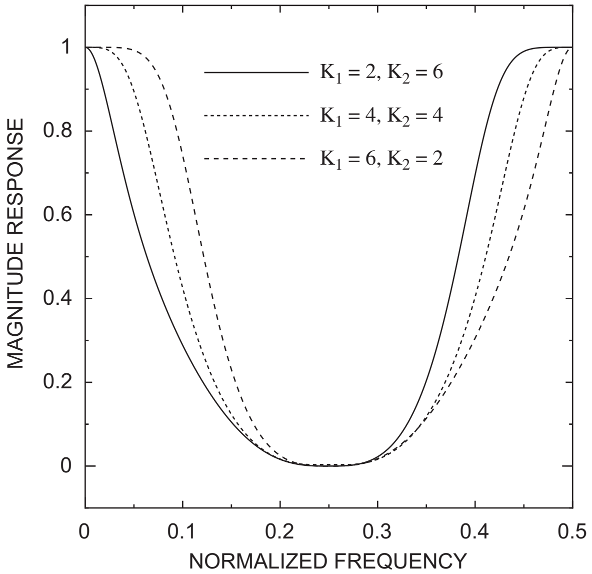

Fig. 8. Magnitude responses of MF IIR bandstop filters in Example 4.

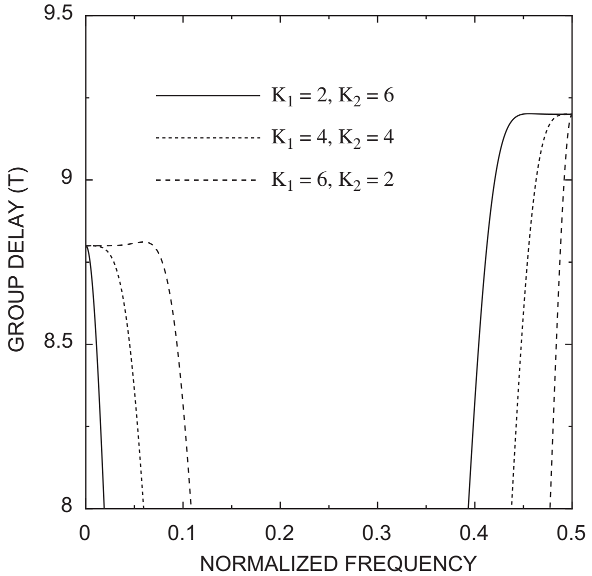

Fig. 9. Group delays of MF IIR bandstop filters in Example 4.

method just needs to solve a system of linear equations, which are derived directly from the flatness conditions given in the passband and stopband. Therefore, a set of filter coefficients can be easily obtained by simply solving this system of linear equations. The feature of the design method is that the flatness of the frequency response can be specified arbitrarily. Moreover, the causality and stability of the proposed MF IIR filters have been examined, and the MF IIR filters become causal and stable when the desired group delay is chosen above a certain value.

Although the design of lowpass, bandpass and bandstop filters have been presented, it is straightforward to extend our design method to various types of digital filters. Moreover, one common problem of MF filters is that the frequency response cannot be directly specified excluding the frequency points on which the conditions of flatness are imposed. Since our method needs to solve a system of linear equations only, it is possible to specify some frequency points, for example, 3dB-points, at the expense of the degree of flatness.

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