ELSEVIER

Signal Processing 80 (2000) 1551–1559

SIGNAL PROCESSING

www.elsevier.nl/locate/sigpro

Design of orthonormal symmetric wavelet filters using real allpass filters

Xi Zhang*, Tomonobu Muguruma, Toshinori Yoshikawa

Department of Electrical Engineering, Nagaoka University of Technology, 1603 - 1 Kamitomioka-machi, Nagaoka, Niigata 940-2188, Japan

Received 13 September 1999; received in revised form 18 January 2000

Abstract

In this paper, a class of real-valued orthonormal symmetric wavelet filters is constructed by using allpass filters, and a new method for designing the allpass-based wavelet filters with the given degrees of flatness is proposed. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm and considering the flatness condition. Therefore, a set of filter coefficients can be easily computed by solving the eigenvalue problem, and the optimal solution in the minimax sense is obtained through a few iterations. Furthermore, the design of the maximally flat allpass-based wavelet filters is also included as a specific case, but it has a closed-form solution that is the same as in Selesnick (IEEE Trans. Signal Process. 46 (4) (April 1998) 1138–1141) so that the iteration procedure is not needed. Finally, some examples are designed to investigate the filter characteristics, and it is shown that the number of delay elements strongly influences the filter magnitude responses. © 2000 Elsevier Science B.V. All rights reserved.

Zusammenfassung

In dieser Arbeit wird die Konstruktion einer Klasse reellwertiger, orthonormaler und symmetrischer Waveletfilter als Allpaßfilter betrachtet. Es wird eine neue Methode für den Entwurf der Allpaß-Waveletfilter mit gewähltem Grad an Glattheit vorgeschlagen. Die vorgeschlagene Methode beruht auf der Formulierung eines verallgemeinerien Eigenwert-problems bei Berücksichtigung der Glattheitsbedingung und der Anwendung des Remezschen Austausch-Algorithms! Deshalb können die Filterkoeffizienten nach Lösung des Eigenwertproblems leicht berechnet werden und die optimale Lösung wird nach wenigen Iterationen erreicht. Der Entwurf des maximal glatten Allpaß-Waveletfilters ist als Spezialfall enthalten. Da seine Lösung in geschlossener Form angegeben werden kann und diese dieselbe ist wie die in Selesnick (IEEE Trans. Signal Process. 46 (4) (April 1998) 1138–1141) vorgeschlagene, ist in diesem Fall die Iterationsprozedur verzichtbar. Es werden einige Beispiel-Wavelets entworfen, um deren Filter-Charakteristik zu untersuchen. Dabei wird gezeigt, daß die Zahl der Verzögerungselemente den Amplitudengang der Filterantwort beeinflußt. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Nous construisons dans cet article une classe de filtres d'ondelettes symétriques, orthonormaux, et à valeurs réelles à l'aides de filtres passe-tout, et présentons une méthode nouvelle de conception des filtres d'ondelettes basés sur les filtres passe-tout et ayant un degré de platitude donné. La méthode proposéee est basée sur la formulation d'un problème de valeurs propres généralisées et utilise l'algorithme d'échange de Remez pour une platitude donnée. De ce fait, un

* Corresponding author. Tel.: + 81-258-47-9522; fax: + 81-258-47-9500.
E-mail address: xiz@nagaokaut.ac.jp (X. Zhang).

PII: S0165-1684(00)00055-4

ensemble de coefficients de filtre peut être aisément calculé en résolvant le problème de valeurs propres, et la solution optimale au sens minimax est obtenue en quelques itérations. De plus, la conception des filtres d'ondelettes maximalement plats est également incluse comme cas particulier, mais elle a une solution analytique identique à celle de Selesnick (IEEE Trans. Signal Process. 46 (4) (April 1998) 1138–1141) et ne requiert done pas de procédure itérative. Enfin, quelques exemples sont conçus pour étudier les caractéristiques des filtres, et il est montré que le nombre d'éléments de retard influence considérablement les résponses en amplitude du filtre.

Keywords: Symmetric wavelets; Allpass filters; Remez exchange algorithm; Eigenvalue problem

1. Introduction

The discrete wavelet transform (DWT), which is implemented by a two-channel perfect reconstruction filter bank (PRFB), has been applied extensively to digital signal and image processing [1–14]. In many applications such as digital image coding, wavelets are required to be real since the signal is real-valued in general. In this paper, we restrict ourselves to real-valued wavelet filters. In addition to orthonormality, one desirable property for wavelets is symmetry, which corresponds to the phase linearity of the wavelet filters. It is known [10] that FIR filters (corresponding to the compactly supported wavelets) can easily realize the linear phase. However, it had been proven in [4] that there does not exist any real-valued compactly supported orthonormal symmetric wavelets except the Haar wavelet. To obtain symmetric wavelets, at least one of the above properties has to be given up. One possible solution to this dilemma is to construct compactly supported biorthogonal symmetric wavelets [4,11,14]. In [2,14], biorthogonal symmetric wavelets have been used in image coding application and are required close to orthonormal. Another possible solution is to construct orthonormal symmetric wavelets by using IIR filters [5,7]. In [5], a class of orthonormal symmetric wavelet filters has been constructed by using real allpass filters, but the design method for these allpass-based wavelet filters is not discussed. A closed-form solution for the maximally flat allpass-based wavelet filters is given in [7], but only the case of $K = 1$ and even N is described, where K is the number of delay elements and N is the order of allpass filter, as is explained in Section 2. It is known [6] that frequency selectivity is a useful property for many applications such as signal processing, but the maximally flat filter has a poor frequency selectivity in general. For this reason, the wavelet filters are required to have the best-possible frequency selectivity for the given degrees of flatness, i.e., the given number of vanishing moments (indication of regularity).

In this paper, we consider the design of the orthonormal symmetric wavelet filters proposed in [5], and propose a new method for designing the allpass-based wavelet filters with the given degrees of flatness. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm and considering the flatness condition. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem [15,16], and the optimal solution in the minimax sense is attained through a few iterations. The proposed design algorithm is computationally efficient because it retains the speed inherent in the Remez exchange algorithm. Furthermore, the design of the maximally flat allpass-based wavelet filters that have the maximal degrees of flatness is also included in the proposed method as a specific case, but it has a closed-form solution that is the same as in [7] so that the iteration procedure is not needed. Finally, we design some examples to investigate the filter characteristics, and show the effects of the number of delay elements on the filter magnitude responses.

This paper is organized as follows. Section 2 describes a class of real-valued orthonormal symmetric wavelet filters composed of allpass filters. Section 3 presents a design method for the allpass-based wavelet filters with the given degrees of flatness based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm. Section 4 shows some design examples to demonstrate the effectiveness of the proposed method, and investigate the filter characteristics. Conclusions are given in Section 5.

2. Orthonormal symmetric wavelet filters

It is well known [3,4] that wavelets can be generated by a two-channel PRFB $\{H(z), G(z)\}$, where $H(z)$ is a lowpass filter and $G(z)$ is highpass. The orthonormal condition that $H(z)$ and $G(z)$ must satisfy is

$$\begin{aligned} &H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1,\\ &G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 1,\\ &\mathrm{H}(z)G(z^{-1}) + H(-z)G(-z^{-1}) = 0. \end{aligned} \tag{1}$$

When symmetric wavelets are required, $H(z)$ and $G(z)$ must have an exact linear phase. In [5], Herley and Vetterli have proposed a class of orthonormal symmetric wavelet filters by using real allpass filters, i.e.,

$$\begin{aligned} &H(z) = \tfrac{1}{2}\{A(z^2) + z^{-K}A(z^{-2})\},\\ &\mathrm{G}(z) = \tfrac{1}{2}\{A(z^2) - z^{-K}A(z^{-2})\}, \end{aligned} \tag{2}$$

where K must be odd to satisfy the orthonormal condition of Eq. (1), and $A(z)$ is an allpass filter of order N and defined by

$$A(z) = z^{-N}\frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}} = \frac{\sum_{n=0}^{N} a_n z^{n-N/2}}{\sum_{n=0}^{N} a_n z^{-n+N/2}}, \tag{3}$$

where a_n is real and $a_0 = 1$. Assume that the phase response of $A(z)$ is $\theta(\omega)$,

$$\begin{aligned} \theta(\omega) &= -N\omega + 2\tan^{-1}\frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}\\ &= 2\tan^{-1}\frac{\sum_{n=0}^{N} a_n \sin(n-N/2)\omega}{\sum_{n=0}^{N} a_n \cos(n-N/2)\omega}. \end{aligned} \tag{4}$$

Then, the frequency responses of $H(z)$ and $G(z)$ are given by

$$\begin{aligned} &H(\mathrm{e}^{\mathrm{j}\omega}) = \mathrm{e}^{-\mathrm{j}(K/2)\omega}\cos\left\{\theta(2\omega) + \frac{K}{2}\omega\right\}\\ &\mathrm{G}(\mathrm{e}^{\mathrm{j}\omega}) = \mathrm{j}\mathrm{e}^{-\mathrm{j}(K/2)\omega}\sin\left\{\theta(2\omega) + \frac{K}{2}\omega\right\}. \end{aligned} \tag{5}$$

It is clear in Eq. (5) that $H(z)$ and $G(z)$ have an exact linear phase, and satisfy the following power-complementary relation:

$$|H(\mathrm{e}^{\mathrm{j}\omega})|^2 + |G(\mathrm{e}^{\mathrm{j}\omega})|^2 = 1. \tag{6}$$

Therefore, the design problem of the allpass-based wavelet filters in Eq. (2) becomes the phase design of allpass filter $A(z)$. However, the design method for these wavelet filters is not discussed in [5]. In [7], a closed-form solution for the maximally flat wavelet filters is given, but only the case of $K = 1$ and even N is described. Now, we will see what happens when $K = 1$ and N is chosen to be odd. We consider the simplest case of $N = 1$. Since there is only one coefficient to be computed, the maximally flat filter design is very easy. The obtained magnitude response of $H(z)$ is shown in Fig. 1 in the solid line. It is clear that $H(z)$ has an undesired zero and bump nearby $\omega = \pi/2$, thus, strictly speaking, it is not a lowpass filter. Here, we attempt to increase the number of delay elements to $K = 3$, i.e.,

$$H(z) = \tfrac{1}{2}\{A(z^2) + z^{-3}A(z^{-2})\}. \tag{7}$$

Its magnitude response is also shown in the dotted line in Fig. 1, and it is clear that it is a reasonable lowpass filter. Therefore, will we ask what value K should be taken to avoid the undesired zero and bump nearby $\omega = \pi/2$? In the following, we will answer this question.

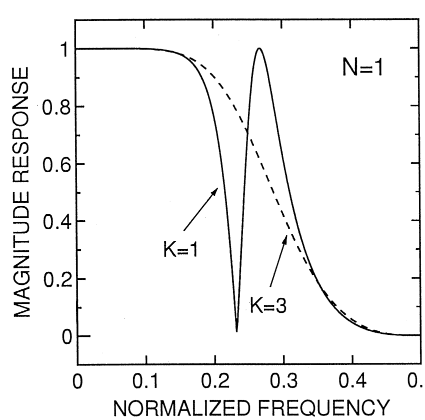

Fig. 1. Magnitude responses of the maximally flat filter $H(z)$ with $K = 1$ and 3.

3. Design of orthonormal symmetric wavelet filters

In this section, we describe the design of the allpass-based orthonormal symmetric wavelet filters with the given degrees of flatness based on a generalized eigenvalue problem by using the Remez exchange algorithm. We have proposed a design method of allpass filters in [16] and will apply it to design the proposed wavelet filters.

3.1. Desired phase response

$H(z)$ and $G(z)$ are required to be a pair of lowpass and highpass filters, and their desired magnitude responses are

$$|H_d(e^{j\omega})| = \begin{cases} 1 & (0 \leqslant \omega \leqslant \omega_p), \\ 0 & (\omega_s \leqslant \omega \leqslant \pi), \end{cases} \tag{8}$$

$$|G_d(e^{j\omega})| = \begin{cases} 0 & (0 \leqslant \omega \leqslant \omega_p), \\ 1 & (\omega_s \leqslant \omega \leqslant \pi), \end{cases} \tag{9}$$

where ω_p and ω_s are the passband and stopband cutoff frequencies of $H(z)$, respectively, and $\omega_p + \omega_s = \pi$. From Eq. (5), the phase response of $A(z)$ must satisfy

$$\theta(2\omega) + \frac{K}{2}\omega = \begin{cases} 0 & (0 \leqslant \omega \leqslant \omega_p), \\ \pm\frac{\pi}{2} & (\omega_s \leqslant \omega \leqslant \pi). \end{cases} \tag{10}$$

Due to the phase antisymmetry, the desired phase response of $A(z)$ can be obtained by

$$\theta_d(\omega) = -\frac{K}{4}\omega \quad (0 \leqslant \omega \leqslant 2\omega_p). \tag{11}$$

Let the phase error be $\theta_e(\omega) = \theta(\omega) - \theta_d(\omega)$,

$$\theta_e(\omega) = 2\tan^{-1}\frac{\sum_{n=0}^{N} a_n \sin(n - N/2 + K/8)\omega}{\sum_{n=0}^{N} a_n \cos(n - N/2 + K/8)\omega} = 2\tan^{-1}\frac{N(\omega)}{D(\omega)}, \tag{12}$$

where

$$N(\omega) = \sum_{n=0}^{N} a_n \sin(n-\tau)\omega,$$

$$D(\omega) = \sum_{n=0}^{N} a_n \cos(n-\tau)\omega \tag{13}$$

and $\tau = N/2 - K/8$. Therefore, the design problem is reduced to the phase approximation of $A(z)$ to the desired phase response of Eq. (11).

3.2. Maximally flat filters

From the regularity of wavelets, $H(z)$ and $G(z)$ are required to meet the following flatness conditions:

$$\left.\frac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0,1,\ldots,2L), \tag{14}$$

$$\left.\frac{\partial^i |G(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0,1,\ldots,2L), \tag{15}$$

where L is integer, and $0 \leqslant L \leqslant N$. This means that $H(z)$ and $G(z)$ contain $(2L+1)$ zeros located at $z = -1$ and 1, respectively. Since $H(z)$ and $G(z)$ satisfy the power-complementary relation of Eq. (6), we consider only the design of $G(z)$ for convenience. Directly using the flatness condition of Eq. (15) will result in a set of nonlinear equations to be solved, which is difficult when L is large. To avoid this difficulty, we decompose $|G(e^{j\omega})|$ of Eq. (5) as

$$|G(e^{j\omega})| = \sin\theta_e(2\omega) = 2\sin\frac{\theta_e(2\omega)}{2}\cos\frac{\theta_e(2\omega)}{2} = 2|G_1(e^{j\omega})||G_2(e^{j\omega})|, \tag{16}$$

where

$$|G_1(e^{j\omega})| = \sin\frac{\theta_e(2\omega)}{2} = \frac{N(2\omega)}{\sqrt{N(2\omega)^2 + D(2\omega)^2}},$$

$$|G_2(e^{j\omega})| = \cos\frac{\theta_e(2\omega)}{2} = \frac{D(2\omega)}{\sqrt{N(2\omega)^2 + D(2\omega)^2}}. \tag{17}$$

By differentiating Eq. (16), we have

$$\frac{\partial^i |G(e^{j\omega})|}{\partial \omega^i} = 2\sum_{k=0}^{i}\binom{i}{k}\frac{\partial^k |G_1(e^{j\omega})|}{\partial \omega^k}\frac{\partial^{i-k} |G_2(e^{j\omega})|}{\partial \omega^{i-k}}. \tag{18}$$

Due to $|G_2(1)| = 1$, thus the flatness condition of Eq. (15) is equivalent to

$$\left.\frac{\partial^i |G_1(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0,1,\ldots,2L). \tag{19}$$

Similarly, from Eq. (17), the condition of Eq. (19) can be reduced to

$$\left.\frac{\partial^i N(\omega)}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i=0,1,\dots,2L). \tag{20}$$

By substituting Eq. (13) into Eq. (20), we get

$$\sum_{n=0}^{N}(n-\tau)^{2i+1}a_n = 0 \quad (i=0,1,\dots,L-1). \tag{21}$$

When the maximally flat filter is needed (i.e., $L=N$), Eq. (21) can be rewritten by using $a_0=1$ as

$$\begin{bmatrix} 1-\tau & 2-\tau & \cdots & N-\tau \\ (1-\tau)^3 & (2-\tau)^3 & \cdots & (N-\tau)^3 \\ \vdots & \vdots & \ddots & \vdots \\ (1-\tau)^{2N-1} & (2-\tau)^{2N-1} & \cdots & (N-\tau)^{2N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \tau \\ \tau^3 \\ \vdots \\ \tau^{2N-1} \end{bmatrix} \tag{22}$$

which can be reduced to a nonsingular Vandermonde matrix. Therefore, there is always a unique solution, and the closed-form formula is given by

$$\begin{aligned} a_n &= (-1)^n\binom{N}{n}\prod_{i=1}^{n}\frac{2\tau-i+1}{2\tau-i-N} \\ &= (-1)^n\binom{N}{n}\prod_{i=1}^{n}\frac{i-1-N+K/4}{i+K/4} \end{aligned} \tag{23}$$

which is the same as in [7].

3.3. Wavelet filters with given flatness

It is well known [9,12] that the maximally flat filter has a poor frequency selectivity. Of course, frequency selectivity is also thought of as a useful property for many applications such as signal processing. However, as is known from Rioul and Duhamel [6], regularity and frequency selectivity somewhat contradict each other. For this reason, we consider the design of the wavelet filters that have the best-possible frequency selectivity for the given regularity, i.e., the given degrees of flatness. When $0 \leqslant L < N$, our aim is to achieve an equiripple response by using the remaining degree of freedom. To obtain an equiripple response, the Remez exchange algorithm is often used, which finds the optimal solution in the minimax sense. It is clear from Eq. (5) that the magnitude response of $G(z)$ satisfies the power-complementary relation between the passband and stopband. Hence, only the stopband response needs to be approximated. First, we select $(N-L+1)$ extremal frequencies ω_i ($\omega_p=\omega_0>\omega_1>\cdots>\omega_{N-L}>0$) in the stopband $[0,\omega_p]$ of $G(z)$. We then use the Remez exchange algorithm and formulate $|G(e^{j\omega})|$ as follows:

$$|G(e^{j\omega_i})| = \sin\theta_e(2\omega_i) = (-1)^i\delta_m, \tag{24}$$

where δ_m (>0) is a magnitude error to be minimized. From Eqs. (12) and (24), we have

$$\frac{\sum_{n=0}^{N}a_n\sin 2(n-\tau)\omega_i}{\sum_{n=0}^{N}a_n\cos 2(n-\tau)\omega_i} = (-1)^i\tan\left\{\frac{\sin^{-1}\delta_m}{2}\right\} = (-1)^i\delta, \tag{25}$$

where $\delta_m = 2\delta/(1+\delta^2) \simeq 2\delta$, and the denominator polynomial must satisfy

$$\sum_{n=0}^{N}a_n\cos(n-\tau)\omega \neq 0 \quad (0\leqslant\omega\leqslant 2\omega_p). \tag{26}$$

Eqs. (21) and (25) can be rewritten in the matrix form as

$$\boldsymbol{P}\boldsymbol{a} = \delta\boldsymbol{Q}\boldsymbol{a}, \tag{27}$$

where $\boldsymbol{a}=[a_0,a_1,\dots,a_N]^T$, and the elements of the matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ are given by

$$P_{ij} = \begin{cases}(j-\tau)^{2i+1} & (i=0,1,\dots,L-1),\\ \sin 2(j-\tau)\omega_{i-L} & (i=L,L+1,\dots,N),\end{cases} \tag{28}$$

$$Q_{ij} = \begin{cases}0 & (i=0,1,\dots,L-1),\\ (-1)^{i-L}\cos 2(j-\tau)\omega_{i-L} & (i=L,L+1,\dots,N).\end{cases} \tag{29}$$

It should be noted that Eq. (27) is a generalized eigenvalue problem, i.e., δ is an eigenvalue and $\boldsymbol{a}$ is a corresponding eigenvector. Therefore, to obtain a solution that satisfies Eq. (26), we only need to find the eigenvector corresponding to the positive minimum eigenvalue, which corresponds to the absolutely minimum one in most cases [15,16]. To achieve an equiripple response, we make use of an iteration procedure to get the optimal solution. The design algorithm is shown as follows.

3.4. Design algorithm

Procedure {Design algorithm of orthonormal symmetric wavelet filters}

Begin

1. Read N, K, L and ω_p.
2. Select the initial extremal frequencies Ω_i $(i = 0,1,\dots,N-L)$ equally spaced in the band $[0, \omega_p]$.

Repeat

1. Set $\omega_i = \Omega_i$ for $i = 0,1,\dots,N-L$.
2. Compute $\boldsymbol{P}$ and $\boldsymbol{Q}$ by using Eqs. (28) and (29), then find the positive minimum eigenvalue of Eq. (27) to obtain a set of filter coefficients a_n.
3. Compute the magnitude response $|G(e^{j\omega})|$ and search the peak frequencies Ω_i in the band $[0, \omega_p]$.

Until Satisfy the following condition for a prescribed small constant ε:

$$\{|\Omega_i - \omega_i| \leqslant \varepsilon \quad (\text{for } i = 0,1,\dots,N-L)\}$$

End.

4. Filter properties and design examples

In this section, we present some design examples to demonstrate the effectiveness of the proposed method, and investigate the filter characteristics, in particular the effects of the number of delay elements on the filter magnitude responses.

Example 1. We consider the design of the minimax wavelet filters with $N = 3$, $L = 0$ and $\omega_p = 0.45\pi$. We designed $A(z)$ with various K by using the proposed method, and found that the range of designable K is $-4N-1 \leqslant K \leqslant 4N+1$. The obtained phase responses of $A(z)$ are shown in Fig. 2. Note that when N increases and/or ω_p decreases, $A(z)$ with a larger $|K|$, such as $|K| = 4N+3$ and so on, can be designed also. It is seen in Fig. 2 that the phase of $A(z)$ at $\omega = \pi$ is $\theta(\pi) = -(N-2k)\pi$, where $0 \leqslant k \leqslant N$. This is because $A(z)$ has k poles located outside the unit circle [16]. The magnitude responses of $H(z)$ with $K = \pm 1$ and ± 3 are shown in Fig. 3, and it is clear that when $K = \pm 1$, $H(z)$ has an undesired zero and bump nearby $\omega = \pi/2$. We recall that the desired phase response of $A(z)$ is $\theta_d(\omega) = -K\omega/4$. When $K = \pm 1$, the phase error at $\omega = \pi$ is $\theta_e(\pi) = \pm 3\pi/4$ due to $\theta_d(\omega) = \pm\,\omega/4$. Since the phase error approximates to 0 in the band $[0, 2\omega_p]$, $\theta_e(\omega)$ changes from 0 to $\pm 3\pi/4$ in the band $[2\omega_p, \pi]$. Thus, there exists a frequency point $\tilde{\omega}$ where $\theta_e(\tilde{\omega}) = \pm\pi/2$, which results in the undesired zero and bump. When $K = 4(N-2k) \pm 3$, the

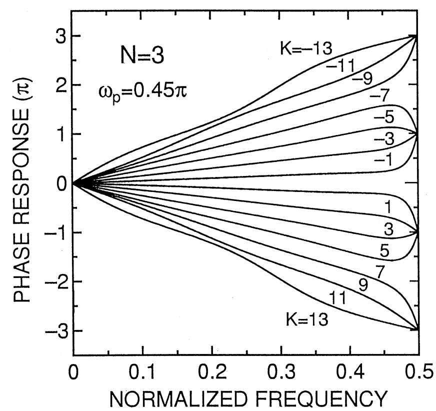

Fig. 2. Phase responses of $A(z)$ with various K in Example 1.

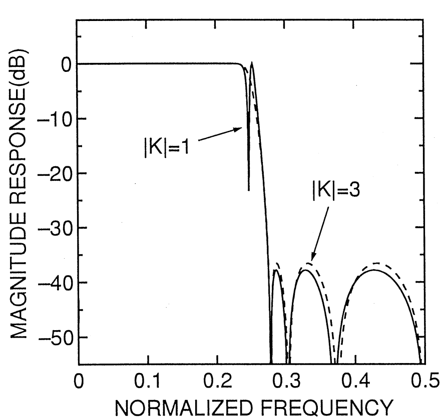

Fig. 3. Magnitude responses of $H(z)$ with $K = \pm 1$ and ± 3 in Example 1.

undesired zero and bump arise similarly due to $\theta_e(\pi) = \pm 3\pi/4$, whereas when $K = 4(N - 2k) \pm 1$, a pair of reasonable lowpass and highpass filters can be obtained. Therefore, to avoid the undesired zero and bump, K must be chosen as $|K| = 3, 5, 11, 13, \ldots$ when N is odd and $|K| = 1, 7, 9, 15, \ldots$ when N is even. It is known from Eq. (5) that the magnitude errors of $H(z)$ and $G(z)$ are dependent on the phase error of $A(z)$. We then investigate the influence of K on the phase error. The plot of the maximum phase error versus K is given in Fig. 4. It is seen that the maximum phase error increases with an increasing $|K|$, thus, we must choose $|K|$ as small as possible to get the minimum error. To summarize the above results, to obtain a pair of reasonable lowpass and highpass filters with the minimum error, the optimal K is $K = \pm 1$ when N is even and $K = \pm 3$ when N is odd.

Example 2. We consider the design of the maximally flat wavelet filters with $N = L = 4$. We designed $A(z)$ with $K = 1$ and 3 by using the proposed method, and its phase responses are shown in Fig. 5. The magnitude responses of $H(z)$ with $K = 1$ and 3 are shown in Fig. 6. As described in Example 1, $H(z)$ of $K = 3$ has the undesired zero and bump nearby $\omega = \pi/2$. The scaling and wavelet functions generated by these wavelet filters are shown in Figs. 7 and 8, respectively. It is seen in Figs. 7 and 8 that the scaling function is symmetric, while the wavelet function is antisymmetric. Although $H(z)$ with $K = 1$ and 3 have the same degrees of flatness, it is

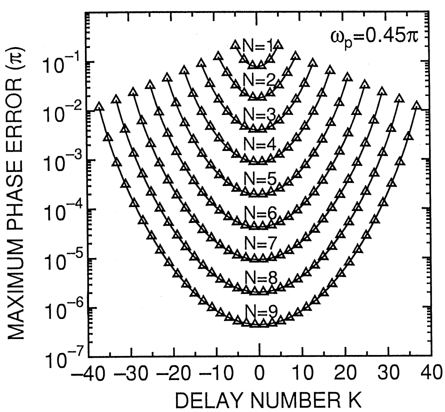

Fig. 4. Maximum phase error versus K in Example 1.

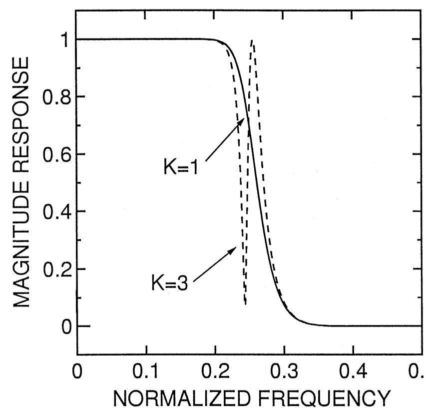

Fig. 6. Magnitude responses of $H(z)$ with $K = 1$ and 3 in Example 2.

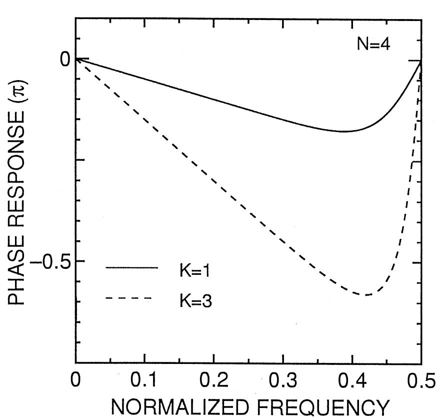

Fig. 5. Phase responses of $A(z)$ with $K = 1$ and 3 in Example 2.

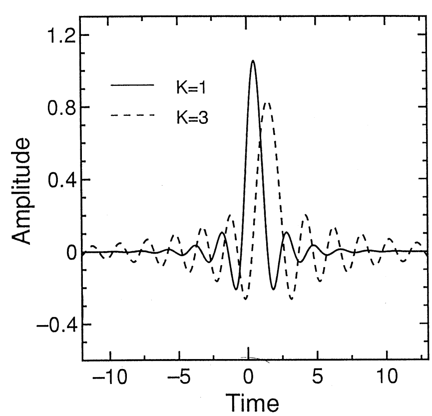

Fig. 7. Scaling function of $K = 1$ and 3 in Example 2.

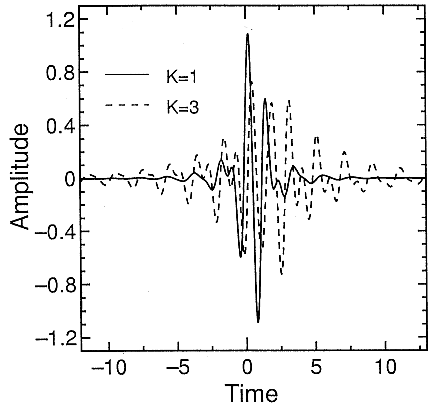

Fig. 8. Wavelet function of $K=1$ and 3 in Example 2.

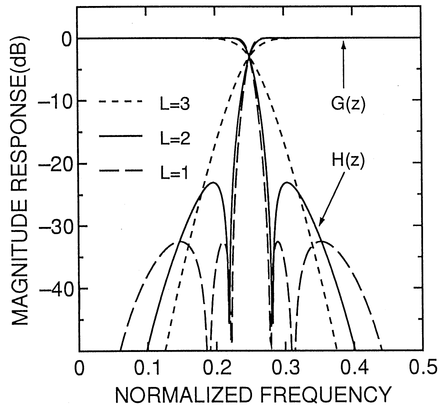

Fig. 10. Magnitude responses of $H(z)$ and $G(z)$ in Example 3.

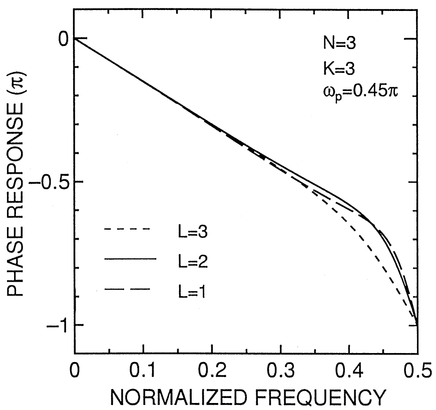

Fig. 9. Phase responses of $A(z)$ in Example 3.

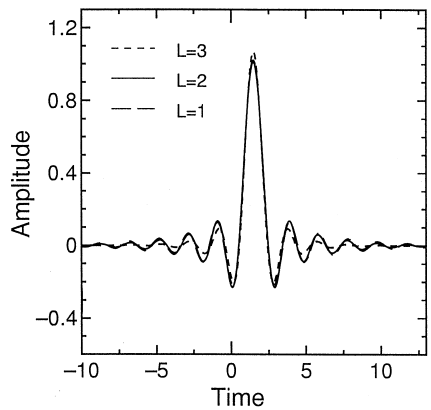

Fig. 11. Scaling functions in Example 3.

seen that the scaling and wavelet functions of $K=3$ decline more slowly than that of $K=1$, because of the influence of the undesired zero and bump.

Example 3. We consider the design of the wavelet filter with $N=3$, $K=3$, and $\omega_p=0.45\pi$. We designed $A(z)$ with $L=3,2,1$ by using the proposed method. The phase responses of $A(z)$ are shown in Fig. 9, and the magnitude responses of $H(z)$ and $G(z)$ are shown in Fig. 10. It is clear in Fig. 10 that the wavelet filter of $L=3$ is maximally flat, and the magnitude error decreases with a decreasing L. The scaling and wavelet functions generated by these wavelet filters are shown in Figs. 11 and 12, respectively. To increase L implies that the resulting wavelet functions are more regular. It is seen in Figs. 11 and 12 that the scaling and wavelet functions decline more rapidly as L increases.

5. Conclusions

In this paper, we have given a class of real-valued orthonormal symmetric wavelet filters composed of allpass filters, and proposed a new method for designing the allpass-based wavelet filters with the given degrees of flatness. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm and considering the flatness condition. Therefore, the filter coefficients can be easily

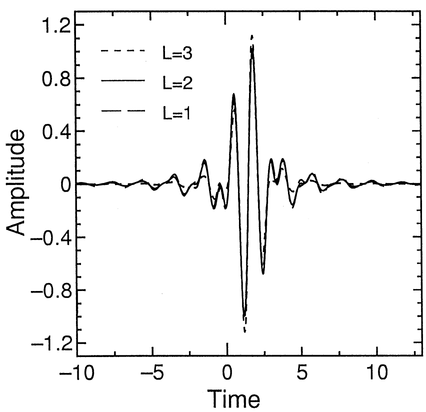

Fig. 12. Wavelet functions in Example 3.

computed by solving the eigenvalue problem, and the optimal solution is obtained through a few iterations. The proposed design algorithm is computationally efficient because it retains the speed inherent in the Remez exchange algorithm. Furthermore, the design of the maximally flat allpass-based wavelet filters is also included as a specific case, but it has a closed-form solution that is the same as in [7]. Finally, it is shown through some design examples that the number of delay elements strongly influences the filter magnitude responses, and must be appropriately chosen to avoid the undesired zero and bump.

References

[1] A.N. Akansu, M.J.T. Smith, Subband and Wavelet Transforms: Design and Applications, Kluwer, Dordrecht, 1996.

[2] M. Antonini, M. Barlaud, P. Mathieu, I. Daubechies, Image coding using wavelet transform, IEEE Trans. Image Process. 1 (2) (April 1992) 205–220.

[3] I. Daubechies, Orthonormal bases of compactly supported wavelets, Commun. Pure Appl. Math. XLI (41) (November 1988) 909–996.

[4] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, PA, 1992.

[5] C. Herley, M. Vetterli, Wavelets and recursive filter banks, IEEE Trans. Signal Process. 41 (8) (August 1993) 2536–2566.

[6] O. Rioul, P. Duhamel, A Remez exchange algorithm for orthonormal wavelets, IEEE Trans. Circuits Systems – II 41 (8) (August 1994) 550–560.

[7] I.W. Selesnick, Formulas for orthogonal IIR wavelet filters, IEEE Trans. Signal Process. 46 (4) (April 1998) 1138–1141.

[8] M.J.T. Smith, S.L. Eddins, Analysis/synthesis techniques for subband image coding, IEEE Trans. Acoust. Speech Signal Process. 38 (8) (August 1990) 1446–1456.

[9] P.P. Vaidyanathan, Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial, Proc. IEEE 78 (1) (January 1990) 56–93.

[10] P.P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[11] M. Vetterli, C. Herley, Wavelets and filter banks: theory and design, IEEE Trans. Signal Process. 40 (9) (September 1992) 2207–2232.

[12] M. Vetterli, J. Kovacevic, Wavelets and Subband Coding, Prentice-Hall, Upper Saddle River, NJ, 1995.

[13] D. Wei, A.C. Bovik, Generalized coiflets with nonzero-centered vanishing moments, IEEE Trans. Circuits Systems – II 45 (8) (August 1998) 988–1001.

[14] D. Wei, J. Tian, R.O. Wells, C.S. Burrus, A new class of biorthogonal wavelet systems for image transform coding, IEEE Trans. Image Process. 7 (7) (July 1998) 1000–1013.

[15] X. Zhang, H. Iwakura, Design of IIR digital filters based on eigenvalue problem, IEEE Trans. Signal Process. 44 (6) (June 1996) 1325–1333.

[16] X. Zhang, H. Iwakura, Design of IIR digital allpass filters based on eigenvalue problem, IEEE Trans. Signal Process. 47 (2) (February 1999) 554–559.