Design of FIR Halfband Filters for Orthonormal Wavelets Using Remez Exchange Algorithm

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***Abstract*—A new method is presented for designing FIR halfband filters for orthonormal wavelets. When the filter degree, number of vanishing moments, and tolerance error in passband/stopband are specified, the filters with the maximal passband/stopband width are designed by using Remez exchange algorithm. It is well-known that Remez exchange algorithm is an efficient approach for equiripple design of FIR linear phase filters. Therefore, FIR halfband filters with a given tolerance error can be easily obtained. Finally, some examples are presented to demonstrate the effectiveness of the proposed design method.**

SPL EDICS : DSP-FILT

***Index Terms*—Equiripple design, FIR halfband filter, orthonormal wavelet, Remez exchange algorithm.**

I. Introduction

IT is known in [1] that orthonormal wavelets can be generated by two band orthonormal filter banks. The key to design two band filter banks is a halfband product filter. Lowpass and highpass filters in the filter banks can be obtained from a spectral factorization of the halfband filter. For wavelet bases to be orthonormal, the frequency response of halfband filters must be nonnegative, that is, the nonnegativity property. It is also known in [1] that the halfband filters are required to have some zeros at $z=-1$, that is, the vanishing moment (VM) condition.

In [1], [5] and [6], the maximally flat FIR halfband filters are used to obtain the maximum number of VMs, that is, the maximum number of zeros are located at $z=-1$. However, the maximally flat filters have a slow transition band roll-off. In constract, Smith and Barnwell used FIR halfband filters with an equiripple (Minimax) magnitude response in [4], which do not preserve any VMs. There are also many methods for designing FIR halfband filters with the specified number of VMs [7]–[12]. In [7], Caglar and Akansu introduced a parametric Bernstein polynomial (PBP) to represent the transfer function of FIR halfband filters. By using this transfer function, the specified number of VMs can be easily obtained only by setting some coefficients to be zero. Therefore, the design becomes how to satisfy the nonnegativity constraint and to make the roll-off sharper simultaneously. In [10], Tay has proposed a simple zero-pinning technique for this problem, but the design is difficult if there are many zeros needed pinning. In [12], Yu pointed out that the zero-pinning technique cannot guarantee the nonnegativity property, thus the frequency response must be both lower- and upper-bounded. Therefore, Yu considered the design problem (*Problem 1* in [12]) in which the passband/stopband width is maximized for a given tolerance error. However, this problem was not directly solved in [12]. Instead, Yu used the semidefinite programming (SDP) to solve another design problem (*Problem 2*) in which the tolerance error is minimized when the passband/stopband is fixed. The solution to *Problem 1* was obtained by solving *Problem 2* iteratively to find an appropriate bandwidth. It is obviously time-consuming.

It is well-known in [2] and [3] that the Remez exchange algorithm is an efficient approach for designing FIR linear phase filters with an equiripple magnitude response. Remez exchange algorithm had been also used to solve *Problem 2* in [8] and [9]. In this letter, we want to use Remez exchange algorithm to solve *Problem 1* directly. Firstly, we describe how to solve *Problem 2* by using Remez exchange algorithm. Next, we propose a new direct method to solve *Problem 1*. In the proposed method, a set of filter coefficients is easily obtained only by solving a system of linear equations. The optimal solution is attained through a few iterations. The proposed design algorithm is computationally efficient because it retains the speed inherent in Remez exchange algorithm. Therefore, FIR halfband filters with a given tolerance error can be easily designed. Finally, some design examples are presented to demonstrate the effectiveness of the proposed design method.

II. Parametric Bernstein Polynomial (PBP)

The PBP was first introduced in [7], and expressed in [10] and [12] as

$$B(x)=K(x)-\sum_{i=L}^{(N-1)/2}\alpha_i K_i(x) \tag{1}$$

where

$$K(x)=\sum_{i=0}^{(N-1)/2}\binom{N}{i}x^i(1-x)^{N-i}, \tag{2}$$

$$K_i(x)=\binom{N}{i}\left[x^i(1-x)^{N-i}-x^{N-i}(1-x)^i\right] \tag{3}$$

and the filter coefficients α_i are real, the filter degree N is odd, and L is integer and $0<L<(N+1)/2$.

By using $x=-z(1-z^{-1})^2/4$, we can obtain the product filter $P(z)=B(-z(1-z^{-1})^2/4)$. It is clear that $P(z)$ is a

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zero-phase halfband filter since $P(z) = P(z^{-1})$ and $P(z) + P(-z) = 1$. In addition, $B(x)$ has L zeros at $x = 1$, that is, $P(z)$ has $2L$ zeros at $z = -1$. It means that L is the number of VMs. If $\alpha_i = 0$ for all i, then the maximally flat FIR filters are obtained, which have the maximum number of VMs, that is, $L_{max} = (N+1)/2$.

Since the VM condition is structurally imposed, the design problem is how to meet the nonnegativity constraint, that is, $B(x) \geq 0$ in [0, 1] for orthonormal wavelets and to make the frequency response sharper simultaneously. It is seen that $B(x) + B(1-x) = 1$. It means that $B(x)$ is antisymmetrical with respect to the point (0.5, 0.5), and $B(0.5) = 0.5$. Thus, the nonnegativity condition is reduced to

$$0 \leq B(x) \leq 1 \quad (0 \leq x < 0.5) \tag{4}$$

or

$$0 \leq B(x) \leq 1 \quad (0.5 < x \leq 1). \tag{5}$$

To have a sharper frequency response, we want $B(x)$ to satisfy

$$1 - \delta \leq B(x) \leq 1 \quad (0 \leq x \leq x_p) \tag{6}$$

and

$$0 \leq B(x) \leq \delta \quad (x_s \leq x \leq 1) \tag{7}$$

where δ $(0 < \delta < 0.5)$ is a tolerance error, and $[0, x_p]$ and $[x_s, 1]$ are the passband and stopband respectively. Because of the antisymmetry of $B(x)$, $x_p + x_c = 1$. The approximation only needs to be done in the passband $[0, x_p]$ or stopband $[x_s, 1]$. In the following, we will consider the design in the stopband $[x_s, 1]$.

In [12], Yu wanted to maximize the passband/stopband width $x_p = 1 - x_s$ for a given tolerance error δ (*Problem 1*). However, it cannot be directly solved by using standard SDP. Instead, Yu used SDP to solve *Problem 2* in which δ is minimized if $x_p = 1 - x_s$ is specified. Then, the solution to *Problem 1* was obtained by solving *Problem 2* iteratively to find an appropriate x_p. It is obviously time-consuming.

III. Design Based on Remez Exchange Algorithm

It is well-known in [2] and [3] that the Remez exchange algorithm is an efficient approach for designing linear phase FIR filters with equiripple frequency responses. In [8] and [9], Remez exchange algorithm had been also used to design FIR halfband filters with the specified number of VMs. In this section, we first describe how to use Remez exchange algorithm to solve *Problem 2*. Next, we propose a new direct method based on Remez exchange algorithm to solve *Problem 1*. Main computation is to solve a system of linear equations, and the optimal solution can be attained through a few iterations. Therefore, the proposed design method is computationally efficient.

By applying Remez exchange algorithm in the stopband $[x_s, 1]$, we first select $(M+1)$ extremal points x_m as $x_s = x_0 < x_1 < \cdots < x_M < 1$, where $M = (N+1)/2 - L$ must be even,[1] as discussed in [8] and [9]. Then, we formulate $B(x)$ as

$$B(x_m) = \frac{1 + (-1)^m}{2}\delta. \tag{8}$$

It should be noted that if x_{2m} for $m = 0, 1, \cdots, M/2$ are maxima and x_{2m-1} for $m = 1, 2, \cdots, M/2$ are minima, then (8) means that (7) is satisfied.

By substituting $B(x)$ in (1) into (8), we derive a system of linear equations as follows;

$$K(x_m) - \sum_{i=L}^{(N-1)/2} \alpha_i K_i(x_m) = \frac{1 + (-1)^m}{2}\delta \tag{9}$$

for $m = 0, 1, \cdots, M$.

If x_s is specified in *Problem 2*, we initially select x_m equally spaced in the stopband $[x_s, 1]$, and then have from (9)

$$\sum_{i=L}^{(N-1)/2} \alpha_i K_i(x_m) + \frac{1 + (-1)^m}{2}\delta = K(x_m) \tag{10}$$

for $m = 0, 1, \cdots, M$. It is clear that there are $(M+1)$ equations with respect to $M = (N+1)/2 - L$ unknown coefficients α_i plus one error δ. Therefore, we can solve (10) to obtain a set of coefficients α_i. Since the maxima x_{2m} and minima x_{2m-1} are unknown *a priori*, we need to utilize an iteration procedure similar to standard Remez exchange algorithm to get the equiripple response of $B(x)$. The design algorithm for *Problem 2* is shown in detail as follows.

Procedure {Design Algorithm for *Problem 2*}

Begin

1) Read N, L, and x_s.
2) Select initial extremal points X_m $(x_s = X_0 < X_1 < \cdots < X_M < 1)$ equally spaced in $[x_s, 1]$.

Repeat

3) Set $x_m = X_m$ for $m = 0, 1, \cdots, M$.
4) Solve (10) to obtain a set of coefficients α_i.
5) Find the maxima X_{2m} and minima X_{2m-1} $(x_s = X_0 < X_1 < \cdots < X_M < 1)$ of $B(x)$ in $[x_s, 1]$.

Until

Satisfy the following condition for a prescribed small constant ϵ (e.g., $\epsilon = 10^{-8}$);

$$\sum_{m=1}^{M} |x_m - X_m| < \epsilon$$

End.

Next, we consider *Problem 1*. When δ is specified, we want to maximize the passband/stopband width $x_p = 1 - x_s$, that is, to minimize x_s. In standard Remez exchange algorithm, $x_0 = x_s$ is also considered as a maximum. However, we cannot select x_m

[1]In [12], Yu restricted that $(N-1)/2 - L$ was even. It is mistaken. $(N+1)/2 - L$ should be even.

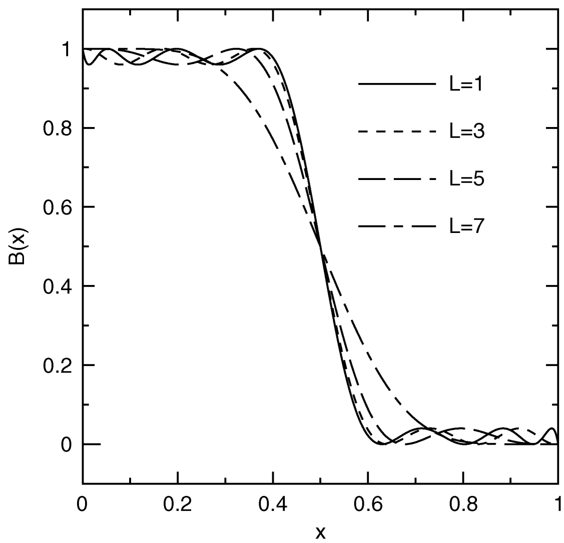

Fig. 1. Responses of $B(x)$ in Example 1.

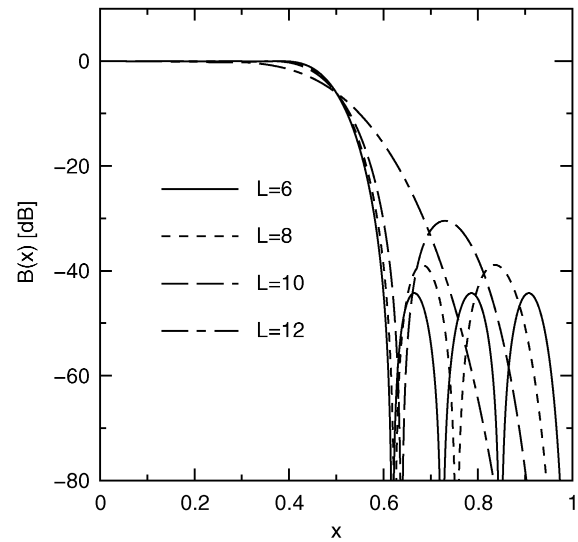

Fig. 2. Responses of $B(x)y$ in Example 2.

in the above-mentioned manner, since x_s is unknown. It should be noted that there are only M unknown α_i in (9), if δ is given. Then, we have

$$\sum_{i=L}^{(N-1)/2} \alpha_i K_i(x_m) = K(x_m) - \frac{1+(-1)^m}{2}\delta. \tag{11}$$

To obtain M unknown α_i, it is known that we only need M equations. Thus, we can exclude $x_0 = x_s$ and select $M/2$ minima x_{2m-1} and $M/2$ maxima x_{2m} for $m = 1, 2, \cdots, M/2$ in such a way that $0.5 < x_1 < x_2 < \cdots < x_M < 1$. By solving (11), we can obtain a set of coefficients α_i such that $B(x_{2m-1}) = 0$ and $B(x_{2m}) = \delta$. Similarly, we find the minima x_{2m-1} and maxima x_{2m}, and then iteratively solve the linear equations in (11) to get the equiripple response. The design algorithm for *Problem 1* is shown as follows.

Procedure {Design Algorithm for *Problem 1*}

Begin

1) Read N, L, and δ.
2) Select initial extremal points X_m $(0.5 < X_1 < X_2 < \cdots < X_M < 1)$ equally spaced in (0.5, 1].

Repeat

3) Set $x_m = X_m$ for $m = 1, 2, \cdots, M$.
4) Solve (11) to obtain a set of coefficients α_i.
5) Find the minima X_{2m-1} and maxima X_{2m} $(0.5 < X_1 < X_2 < \cdots < X_M < 1)$ of $B(x)$ in (0.5, 1].

Until

Satisfy the following condition for a prescribed small constant ϵ (e.g., $\epsilon = 10^{-8}$);

$$\sum_{m=1}^{M} |x_m - X_m| < \epsilon$$

End.

IV. Design Examples

In this section, we present two design examples to demonstrate the effectiveness of the proposed design method.

Example 1: We consider the design of FIR halfband filters with $N = 13$ in [12]. The tolerance error is specified to be $\delta = 0.04$, and the number of VMs is chosen as $L = 7$, 5, 3, 1. $L = 7$ is correspondent to the maximally flat filter. The filter coefficients are easily obtained by using the design algorithm for *Problem 1* with 5 iterations, and the resulting responses are shown in Fig. 1, which are the same as in Fig. 3 of [12].[2] It is seen in Fig. 1 that $1 - \delta \le B(x) \le 1$ in passband and $0 \le B(x) \le \delta$ in stopband, and the band width increases as L decreases.

Example 2: We consider the design of FIR halfband filters with $N = 23$. The band width is specified to be $x_p = 0.4$ and $x_s = 0.6$. The number of VMs is chosen as $L = 12$, 10, 8, 6. $L = 12$ is correspondent to the maximally flat filter. This class of filters are designed by using the design algorithm for *Problem 2*. The resulting responses are shown in dB in Fig. 2. It is seen in Fig. 2 that the error δ decreases as L decreases. The number of iteration is 5.

V. Conclusion

In this letter, we have proposed a new design method of FIR halfband filters for orthonormal wavelets by using Remez exchange algorithm. In the proposed method, a set of filter coefficients can be easily obtained only by solving a system of linear equations, and the optimal solution is attained through a few iterations. Therefore, the proposed design algorithm is computationally efficient. The nonnegativity property for orthonormal wavelets is ensured by forcing $0 \le B(x) \le \delta$ in the stopband. By adjusting $-\delta \le B(x) \le \delta$ in the stopband, the design algorithm is also applicable to biorthogonal case where the nonnegativity is not needed [11]. Finally, some design examples are presented to demonstrate the effectiveness of the proposed design method.

[2]In Fig. 3 of [12], X-axis should be ω/π, not x, where $\omega = 2\sin^{-1}(\sqrt{x})$.

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