A New Phase-Factor Design Method for Hilbert-Pairs of Orthonormal Wavelets

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***Abstract*—A new method is proposed for designing Hilbert transform pairs of orthonormal wavelet bases with improved analyticity. Selesnick proposed a simple common factor technique for designing the Hilbert transform pairs in [7], where the phase factor is required to satisfy the half-sample delay condition, while the common factor is used to obtain the maximum number of vanishing moments and to satisfy the condition of orthonormality. To improve the analyticity of complex wavelets, we propose a novel method to design the phase factor by using the Remez exchange algorithm, so that the difference in the frequency response between two scaling lowpass filters is minimized. One design example is presented to demonstrate the effectiveness of the proposed method.**

***Index Terms*—Analyticity, Hilbert transform pair, orthonormal wavelet, Remez exchange algorithm, vanishing moment.**

I. Introduction

HILBERT transform pairs of wavelets have been proposed and found to be successful in many applications [3]–[12]. It has been proven in [6], [8], [9] that the half-sample delay condition between two scaling lowpass filters is the necessary and sufficient condition for the corresponding wavelet bases to form a Hilbert transform pair. Many design methods for the Hilbert transform pairs have been presented in [3]–[7], [11], [12]. In [7], Selesnick had proposed a simple common factor technique, where the common factor is used to satisfy the condition of orthonormality and to obtain the maximum number of vanishing moments, while the phase factor is required to meet the half-sample delay condition. In [7], Selesnick had used the maximally flat phase approximation for the phase factor. However, the maximally flat approximation yields a larger phase error as $|\omega|$ increases, thus it will influence the analyticity of complex wavelets. In [11], we have improved the analyticity by using the Remez exchange algorithm to sharpen the magnitude responses of scaling lowpass filters, at the expense of vanishing moments. In [12], Tay has presented a downsampling-based approach for designing the phase factor to increase the sharpness of the magnitude responses.

In this letter, we propose a new method for designing the phase factor by using the Remez exchange algorithm, where the difference in the frequency response between two scaling lowpass filters is minimized to improve the analyticity of complex wavelets. It is well-known in [2] that the Remez exchange algorithm is an efficient approach for designing FIR linear phase filters with an equiripple magnitude response. In the proposed method, the design problem of the phase factor is reduced to the design of FIR linear phase filters, thus, a set of filter coefficients can be easily obtained only by using the Remez exchange algorithm. The optimal solution is attained through a few iterations. Therefore, the proposed design algorithm is computationally efficient. Finally, one design example is presented and compared with the conventional methods to demonstrate the effectiveness of the proposed method.

II. Hilbert Transform Pair of Wavelet Bases

It is known in [1] that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z), G_i(z)\}$, where $i = 1, 2$. Now we assume that $H_i(z)$ and $G_i(z)$ are lowpass and highpass filters, respectively. The orthonormality condition of two-band filter banks $\{H_i(z), G_i(z)\}$ are given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0. \end{cases} \tag{1}$$

We denote the scaling and wavelet functions by $\phi_i(t), \psi_i(t)$ respectively. Thus, the corresponding dilation and wavelet equations are expressed as

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n) \end{cases} \tag{2}$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively.

It is known in [6], [8] and [9] that two wavelet functions are a Hilbert transform pair:

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\} \tag{3}$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases} \tag{4}$$

if and only if two scaling lowpass filters satisfy the following condition;

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \quad (-\pi < \omega < \pi) \tag{5}$$

where $\Psi_i(\omega)$ is the Fourier transform of $\psi_i(t)$. Equation (5) is the so-called half-sample delay condition between two scaling lowpass filters. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. It is the necessary

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and sufficient condition for two wavelet bases to form a Hilbert transform pair.

It is known that the complex wavelet function $\psi^c(t) = \psi_1(t) + j\psi_2(t)$ is analytic, i.e., its spectrum is one-sided: $\Psi^c(\omega) = \Psi_1(\omega) + j\Psi_2(\omega) = 0$ for $\omega < 0$, if $\psi_1(t)$ and $\psi_2(t)$ are an ideal Hilbert transform pair. However, it cannot be exact in practice, because the half-sample delay condition (5) can only be approximated with realizable filters. To evaluate the analyticity, we use the p-norm of the spectrum $\Psi^c(\omega)$ to define an objective measure of quality as

$$E_p = \frac{\|\Psi^c(\omega)\|_{p,[-\infty,0]}}{\|\Psi^c(\omega)\|_{p,[0,\infty]}} \tag{6}$$

where

$$\|\Psi^c(\omega)\|_{p,\Omega} = \left(\int_\Omega |\Psi^c(\omega)|^p \, d\omega\right)^{\frac{1}{p}}. \tag{7}$$

If $p = \infty$, $E_\infty = \lim_{p\to\infty} E_p$ is the peak error in the negative frequency domain, which is equal to E_1 proposed in [12]. When $p = 2$, E_2 is the square root of the negative frequency energy.[1] In this letter, we will use E_∞ and E_2 defined in (6) to evaluate the analyticity of the complex wavelet functions.

III. The Common Factor Technique

In [7], Selesnick had proposed a common factor technique for Hilbert transform pairs of orthonormal wavelet bases. The scaling lowpass filters $H_1(z)$ and $H_2(z)$ are constructed by

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}) \end{cases} \tag{8}$$

where $F(z)$ is the common factor, and $D(z)$ is the phase factor and is given by

$$D(z) = \sum_{n=0}^{L} d(n)z^{-n} \tag{9}$$

where L is the degree of $D(z)$, $d(n)$ are real filter coefficients and $d(0) = 1$.

By defining the transfer function of the allpass filter $A(z)$ as

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)} \tag{10}$$

it can be easily verified that

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z) \tag{11}$$

that is, $H_2(z)$ can be expressed as the product of $H_1(z)$ and $A(z)$. If the allpass filter $A(z)$ is an approximate half-sample delay:

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}} \quad (-\pi < \omega < \pi) \tag{12}$$

then the half-sample delay condition (5) is achieved approximately, and thus two orthonormal wavelet bases form an approximate Hilbert transform pair.

Once $A(z)$ is determined, $F(z)$ needs to be designed for $H_1(z)$ and $H_2(z)$. To obtain wavelet bases with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1 + z^{-1})^K. \tag{13}$$

Thus

$$\begin{cases} H_1(z) = Q(z)(1 + z^{-1})^K D(z) \\ H_2(z) = Q(z)(1 + z^{-1})^K z^{-L}D(z^{-1}). \end{cases} \tag{14}$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$:

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(1+z)^K(1+z^{-1})^K D(z)D(z^{-1}). \end{aligned} \tag{15}$$

Let $Q(z)$ be an FIR filter and defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-R}^{R} r(n)z^{-n} \tag{16}$$

$$\begin{aligned} S(z) &= (z + 2 + z^{-1})^K D(z)D(z^{-1}) \\ &= \sum_{n=-L-K}^{L+K} s(n)z^{-n} \end{aligned} \tag{17}$$

where $r(n) = r(-n)$ for $1 \le n \le R$ and $s(n) = s(-n)$ for $1 \le n \le L + K$, then we have

$$P(z) = R(z)S(z). \tag{18}$$

We can write the orthonormality condition (1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} 1 & (n = 0) \\ 0 & \left(1 \le n \le \frac{R+L+K}{2}\right) \end{cases} \tag{19}$$

where $I_{min} = \max\{-R, 2n - L - K\}$ and $I_{max} = \min\{R, 2n + L + K\}$. Note that $P(z)$ is a halfband filter, thus the degree of $H_i(z)$ is $M = R + L + K$ and is an odd number. Since $r(n) = r(-n)$, there are $(M + 1)/2$ equations with respect to $(R + 1)$ unknown coefficients $r(n)$ in (19). Therefore, it is clear that we can obtain a unique solution if $(M + 1)/2 = R + 1$. In [7], Selesnick had chosen $R = L + K - 1$ and obtained the filter of minimal degree for given L and K, which corresponds to the maximal K ($K_{max} = R - L + 1 = (M + 1)/2 - L$) for given L and R. Thus the scaling lowpass filters have the maximally flat magnitude response, resulting in the maximum number of vanishing moments.

IV. Phase Factor Design Using Remez Exchange Algorithm

Now, we define the error function $E(\omega)$ as

$$E(\omega) = H_2(e^{j\omega}) - H_1(e^{j\omega})e^{-j\frac{\omega}{2}}. \tag{20}$$

[1]The negative frequency energy was defined as E_2 in [12].

From (11), we have

$$E(\omega) = H_1(e^{j\omega})\left[A(e^{j\omega}) - e^{-j\frac{\omega}{2}}\right] \tag{21}$$

thus the magnitude response of $E(\omega)$ is

$$|E(\omega)| = 2\left|H_1(e^{j\omega})\right| \left|\sin\left(\frac{\theta(\omega)}{2} + \frac{\omega}{4}\right)\right| \tag{22}$$

where $\theta(\omega)$ is the phase response of $A(z)$. It is clear that the magnitude response of $E(\omega)$ is dependent on both the magnitude response of $H_1(z)$ and the phase error of $A(z)$. Since $H_1(z)$ is a lowpass filter, we must minimize the phase error not only in passband but also in transition band of $H_1(z)$. In [7], Selesnick had chose the maximally flat allpass filters. Since $\omega = 0$ is chosen as the point of approximation, the phase error will increase as increasing $|\omega|$. Thus, $|E(\omega)|$ has a large error in transition band, as shown in [11].

In the following, we will discuss how to design the phase factor to improve the analyticity. From (20), we have

$$\begin{aligned} E(\omega) &= F(e^{j\omega})\left[e^{-jL\omega}D(e^{-j\omega}) - D(e^{j\omega})e^{-j\frac{\omega}{2}}\right] \\ &= 2jF(e^{j\omega})e^{-j\left(\frac{L}{2}+\frac{1}{4}\right)\omega} \\ &\quad \times \sum_{n=0}^{L} d(n)\sin\left(\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega\right) \end{aligned} \tag{23}$$

and then define

$$E_A(\omega) = 2\left|F(e^{j\omega})\right| \sum_{n=0}^{L} d(n)\sin\left(\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega\right). \tag{24}$$

Thus, it is clear that $|E(\omega)| = |E_A(\omega)|$. $E_A(\omega)$ is linear with respect to $d(n)$, if $|F(e^{j\omega})|$ is known. Therefore, it can be reduced to the design of FIR linear phase filters, where $|F(e^{j\omega})|$ is viewed as a weighting function.

Next, we use the Remez exchange algorithm in [2] to obtain an equiripple response of $E(\omega)$. Let $\omega_i(0 < \omega_0 < \omega_1 < \cdots < \omega_L < \pi)$ be the extremal frequencies. We formulate $E_A(\omega)$ as

$$\begin{aligned} E_A(\omega_i) &= 2\left|F(e^{j\omega_i})\right| \sum_{n=0}^{L} d(n)\sin\left(\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega_i\right) \\ &= (-1)^i\delta \end{aligned} \tag{25}$$

where δ is an error. Then, we rewrite (25) as

$$\sum_{n=0}^{L} \frac{d(n)}{\delta}\sin\left(\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega_i\right) = \frac{(-1)^i}{2\left|F(e^{j\omega_i})\right|}. \tag{26}$$

In the common factor technique in [7], $D(z)$ is firstly designed by using the maximally flat approximation, then $F(z)$ is obtained by solving the linear equations in (19) and using the spectral factorization. Thus, $F(z)$ is unknown before $D(z)$ is designed.

In this letter, we use $\tilde{F}(z)$ instead of $F(z)$, and assume $\tilde{d}(n) = d(n)/\delta$, then (26) becomes

$$\sum_{n=0}^{L} \tilde{d}(n)\sin\left(\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega_i\right) = \frac{(-1)^i}{2\left|\tilde{F}(e^{j\omega_i})\right|} \tag{27}$$

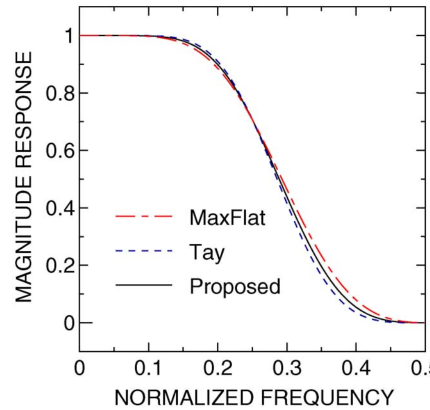

Fig. 1. Magnitude responses of scaling lowpass filters $H_i(z)$.

where $\tilde{F}(z)$ is designed by using the method proposed in [7]. Since $\tilde{F}(z)$ is known, we can view $|\tilde{F}(e^{j\omega})|$ as the weighting function in the FIR linear phase filter design, and then solve the linear equations in (27) to obtain a set of filter coefficients $\tilde{d}(n)$. Note that since $d(0) = 1$, $\delta = 1/\tilde{d}(0)$, and $d(n) = \tilde{d}(n)/\tilde{d}(0)$ for $n = 1, 2, \cdots, L$. Moreover, we make use of an iteration procedure to obtain an equiripple response. The design algorithm is shown as follows.

Procedure {Phase Factor Design Algorithm}

Begin

1) Read L, and $|\tilde{F}(e^{j\omega})|$.
2) Select initial extremal frequencies $\Omega_i(0 < \Omega_0 < \Omega_1 < \cdots < \Omega_L < \pi)$ equally spaced in $(0, \pi)$.

Repeat

3) Set $\omega_i = \Omega_i$ $(i = 0, 1, \cdots, L)$.
4) Solve (27) to obtain a set of filter coefficients $d(n)$.
5) Search the peak frequencies $\Omega_i(0 < \Omega_0 < \Omega_1 < \cdots < \Omega_L < \pi)$ of $E_A(\omega)$ in $(0, \pi)$.

Until

Satisfy the following condition for a prescribed small constant ϵ;

$$|\omega_i - \Omega_i| < \epsilon \qquad \text{(for all } i\text{)}$$

End.

V. Design Example

In this section, we present one design example and compare the frequency responses with the conventional methods to demonstrate the effectiveness of the proposed design method.

We consider the Hilbert transform pair of orthonormal wavelet bases with $K = 2$, $L = 4$, $R = 5$ in [12, Example 1]. The degree of $H_i(z)$ is $M = 11$. Firstly, we designed $H_i(z)$ by using the method proposed in [7]. Then, using the obtained common factor as $\tilde{F}(z)$, we have designed $D(z)$ by the above-mentioned phase factor design algorithm, and obtained $H_i(z)$. The resulting magnitude response of $H_i(z)$ is shown in solid line in Fig. 1. For comparison, the magnitude responses of $H_i(z)$ in [7] and in [12] by Tay are shown in Fig. 1 also. Note that the maximally flat (MaxFlat) allpass filter was used in [7]. It is seen in Fig. 1 that the magnitude response of the

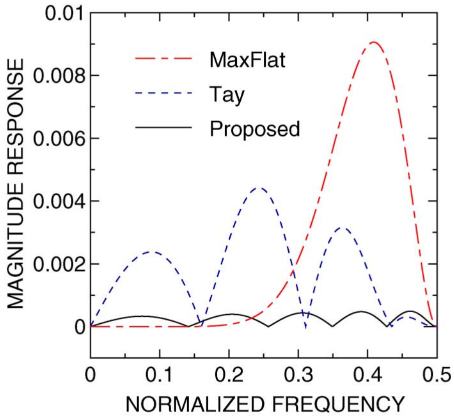

Fig. 2. Magnitude responses of $E(\omega)$.

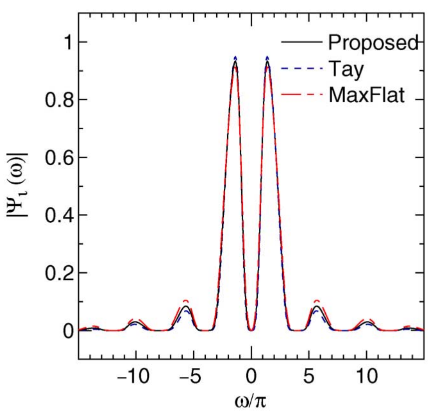

Fig. 3. Magnitude responses of $\Psi_i(\omega)$.

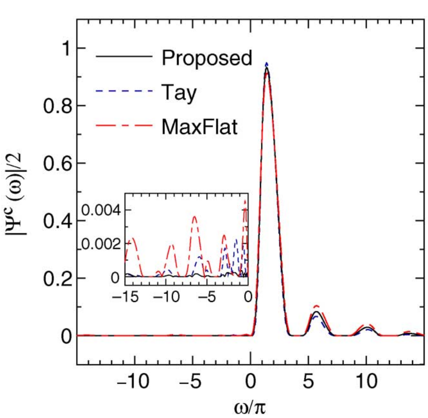

Fig. 4. Magnitude responses of $\Psi^c(\omega)$.

proposed filter is between the conventional filters. Moreover, the magnitude responses of $E(\omega)$ are shown in Fig. 2, and the maximum error of the proposed filter is the smallest. Finally, the spectrums $\Psi_i(\omega)$ of $\psi_i(\omega)$ and $\Psi^c(\omega)$ of $\psi^c(\omega)$ are shown in Fig. 3 and Fig. 4, respectively. It is clear in Fig. 4 that the spectrum $\Psi^c(\omega)$ using the proposed filter is closest to zero in the negative frequency domain ($\omega < 0$), although the spectrums $\Psi_i(\omega)$ are almost same in Fig. 3. Analyticity measures are given in Table I, and both E_∞ and E_2 of the proposed filter are the smallest.

Discussion: In the proposed algorithm, we have used $\tilde{F}(z)$ instead of $F(z)$, thus the resulting $E(\omega)$ is not exactly equiripple, as shown in Fig. 2. It is possible to repeat the proposed algorithm using the obtained $F(z)$ as $\tilde{F}(z)$ to get the equiripple response of $E(\omega)$. However, it cannot be guaranteed to obtain a further improvement of the analyticity, because the wavelet function is defined by the infinite product formula.

TABLE I
ANALYTICITY MEASURES OF E_∞ AND E_2

	$E_\infty(\%)$	$E_2(\%)$
Proposed	0.0424	0.0487
Tay [12]	0.2403	0.2389
MaxFlat [7]	0.4955	0.6250

VI. CONCLUSION

In this letter, we have proposed a new method for designing Hilbert transform pairs of orthonormal wavelet bases with improved analyticity. To improve the analyticity of complex wavelets, we have used the well-known Remez exchange algorithm to design the phase factor, so that the difference in the frequency response between two scaling lowpass filters is minimized. Since the design problem of the phase factor has been reduced to the design of an FIR linear phase filter, a set of filter coefficients can be easily obtained by iteratively solving a system of linear equations, and the optimal solution is attained through a few iterations. Therefore, the proposed design algorithm is computationally efficient. Finally, one design example has been presented and compared with the conventional methods to demonstrate the effectiveness of the proposed method.

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