

Hilbert Pair of Almost Symmetric Orthogonal Wavelets with Arbitrary Center of Symmetry

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Abstract—This paper proposes a new method for designing a class of Hilbert pairs of almost symmetric orthogonal wavelets with arbitrary center of symmetry. Two scaling lowpass filters are designed simultaneously to satisfy the specified degree of flatness of group delays, vanishing moments and orthogonality condition of wavelets, along with improved analyticity. Therefore, the resulting scaling lowpass filters have flat group delay responses and the specified number of vanishing moments. Moreover, the difference of the frequency responses between two scaling lowpass filters can be effectively minimized to improve the analyticity of complex wavelets. The condition of orthogonality is linearized, and then an iterative procedure is used to obtain the filter coefficients. Finally, several examples are presented to demonstrate the effectiveness of the proposed design procedure.

I. INTRODUCTION

The Dual Tree Complex Wavelet Transform (DTCWT) was originally proposed by Kingsbury in [2], and has been found to be successful in a variety of applications of signal processing and image processing [2]~[11]. DTCWT has limited computational redundancy compared with the conventional undecimated wavelets, while it is of approximate shift invariance, and possesses a good directional selectivity for multidimensional signals [2]. The key of DTCWT lies in that two scaling lowpass filters are required to satisfy the half-sample delay condition, resulting in the corresponding wavelet bases form a Hilbert transform pair.

Several design procedures for DTCWT had been presented in [2]~[15]. In [6], Selesnick had proposed a common-factor design technique based on the maximally flat allpass filters. This method is simple and effective, but the resulting filters have non-linear phase responses. In [3] and [15], Q-shift filter was firstly introduced by Kingsbury in order to provide the improved orthogonality and symmetry properties. Q-shift filters are required to have linear phase responses. The design technique proposed in [3] and [4] was based on the optimization of a set of rotations θ_i in the polyphase structure, but this is a highly non-linear problem and works well for relatively short filters. An alternative method proposed by Kingsbury in [7] works effectively for Q-shift filters of length up to 50 or more taps. In [13], Zhang had proposed a design method of Q-shift filters with improved vanishing moments based on the condition of flatness of group delay, vanishing moments and orthogonality. The resulting Q-shift filters have flat group delay responses and the specified number of vanishing moments. Moreover, Zhang and Morihara made

use of the transfer function proposed by Gopinath in [9] that satisfies the flatness condition of group delay at $\omega = 0$ and the number of vanishing moments at $z = -1$. Thus a set of equations is derived only from the condition of orthonormality in [14].

In this paper, we propose a new method for designing a class of Hilbert pairs of almost symmetric orthogonal wavelets with arbitrary center of symmetry. Differently from the technique proposed in [13], the scaling lowpass filters have arbitrarily specified linear phase responses. We specify the degree of flatness for the group delay responses at $\omega = 0$, then locate the specified number of zeros at $z = -1$ from the viewpoint of vanishing moments. Moreover, the difference of the frequency responses between two scaling lowpass filters are minimized to improve the analyticity of DTCWTs. Therefore, the resulting scaling lowpass filters have flat group delay responses and the specified number of vanishing moments. In the proposed method, the filter coefficients of two lowpass filters can be obtained simultaneously by iteratively solving a set of linear equations only. Finally, several examples are presented to demonstrate the effectiveness of the proposed design method.

II. HILBERT PAIR OF ALMOST SYMMETRIC ORTHOGONAL WAVELETS

It is well-known that DTCWT is constituted of two real discrete wavelet transforms (DWTs), where the first DWT gives the real part of DTCWT and the other one is the imaginary part. Let $\phi_H(t)$, $\phi_G(t)$ and $\psi_H(t)$, $\psi_G(t)$ be the scaling and wavelet functions of two DWTs, respectively. It has been shown in [5], [8] and [10] that two wavelet functions $\psi_H(t)$ and $\psi_G(t)$ form a Hilbert transform pair:

$$\psi_G(t) = \mathcal{H}\{\psi_H(t)\}, \quad (1)$$

that is

$$\Psi_G(\omega) = \begin{cases} -j\Psi_H(\omega) & (\omega > 0) \\ j\Psi_H(\omega) & (\omega < 0) \end{cases}, \quad (2)$$

where $\Psi_H(\omega)$ and $\Psi_G(\omega)$ are the Fourier transforms of $\psi_H(t)$, $\psi_G(t)$, respectively.

It is known that if two wavelet functions are a pair of Hilbert transform, the complex wavelet $\psi_c(t) = \psi_H(t) + j\psi_G(t)$ is ideally analytic, i.e., the spectrum is one-sided:

$$\Psi_c(\omega) = \Psi_H(\omega) + j\Psi_G(\omega) = \begin{cases} 2\Psi_H(\omega) & (\omega > 0) \\ 0 & (\omega < 0) \end{cases}. \quad (3)$$

In [5], Selesnick had proved that the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair is that the corresponding scaling lowpass filters satisfy

$$G(e^{j\omega}) = H(e^{j\omega})e^{-j\frac{\omega}{2}} \quad (|\omega| < \pi). \quad (4)$$

Eq.(4) is the so-called half-sample delay condition between two scaling lowpass filters $H(z)$ and $G(z)$. Specifically, the scaling lowpass filters should be offset from one another by a half sample. Since $G(e^{j\omega})$ needs to be approximated to $H(e^{j\omega})e^{-j\frac{\omega}{2}}$, we define the error function $E(\omega)$ between $H(e^{j\omega})$ and $G(e^{j\omega})$,

$$E(\omega) = G(e^{j\omega}) - H(e^{j\omega})e^{-j\frac{\omega}{2}}. \quad (5)$$

Moreover, to evaluate the analyticity of corresponding complex wavelets, we use the p -norm of the spectrum $\Psi_c(\omega)$ to define an objective measure of quality as

$$E_p = \frac{\|\Psi_c(\omega)\|_{p,[-\infty,0]}}{\|\Psi_c(\omega)\|_{p,[0,\infty]}}, \quad (6)$$

where

$$\|\Psi_c(\omega)\|_{p,\Omega} = \left(\int_{\Omega} |\Psi_c(\omega)|^p d\omega \right)^{\frac{1}{p}}. \quad (7)$$

If $p = \infty$, $E_{\infty} = \lim_{p \rightarrow \infty} E_p$ is the peak error in the negative frequency domain. If $p = 2$, E_2 is the square root of the negative frequency energy. In this paper, we will use E_{∞} and E_2 to evaluate the analyticity of the complex wavelet.

The transfer functions of FIR filters $H(z)$, $G(z)$ are given by

$$\begin{cases} H(z) = \sum_{n=0}^N h(n)z^{-n} \\ G(z) = \sum_{n=0}^N g(n)z^{-n} \end{cases}, \quad (8)$$

where $h(n)$, $g(n)$ are real filter coefficients and the degree N is an odd number. In addition, $H(z)$ and $G(z)$ have to satisfy the following condition of orthogonality to generate the orthonormal wavelet bases;

$$\begin{cases} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2 \\ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2 \end{cases}. \quad (9)$$

Furthermore, the lowpass filters are required to have linear phase responses, therefore the desired phase responses of $H(z)$, $G(z)$ are

$$\begin{cases} \theta_d^H(\omega) = -\tau_0\omega \\ \theta_d^G(\omega) = -(\tau_0 + \frac{1}{2})\omega \end{cases}, \quad (10)$$

where the group delay τ_0 can be arbitrarily specified. Thus, the orthogonal wavelets are almost symmetric and have an arbitrary center of symmetry by varying the group delay τ_0 .

III. DESIGN OF ALMOST SYMMETRIC ORTHOGONAL WAVELET FILTERS

In this section, we discuss the design of orthogonal wavelet filter $H(z)$ with flat group delay and the specified number of vanishing moments. It should be noted that $G(z)$ can be designed similarly with the group delay $(\tau_0 + 1/2)$.

Firstly, we consider the flatness condition of the group delay response. There exist many criterions in the group delay approximations, e.g., weighted least square, equiripple approximation, and so on [1]. In this paper, we consider the maximally flat approximation. From Eq.(8), the phase response of $H(z)$ is given by

$$\theta(\omega) = -\tan^{-1} \frac{\sum_{n=0}^N h(n) \sin(n\omega)}{\sum_{n=0}^N h(n) \cos(n\omega)}. \quad (11)$$

Thus, the difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d^H(\omega)$ is

$$\theta_e(\omega) = \theta(\omega) - \theta_d^H(\omega) = \tan^{-1} \frac{N(\omega)}{D(\omega)}, \quad (12)$$

where

$$\begin{cases} N(\omega) = \sum_{n=0}^L h(n) \sin\{(\tau_0 - n)\omega\} \\ D(\omega) = \sum_{n=0}^L h(n) \cos\{(\tau_0 - n)\omega\} \end{cases}. \quad (13)$$

$H(z)$ is required to have the specified degree of flatness at $\omega = 0$ for the group delay response,

$$\begin{cases} \tau(0) = \tau_0 \\ \left. \frac{\partial^{2r} \tau(\omega)}{\partial \omega^{2r}} \right|_{\omega=0} = 0 \quad (r = 1, 2, \dots, L-1) \end{cases}, \quad (14)$$

where $L (> 0)$ controls the degree of flatness. Since $\tau(\omega) = -\frac{\partial \theta(\omega)}{\partial \omega}$, Eq.(14) is equivalent to

$$\left. \frac{\partial^{2r+1} \theta_e(\omega)}{\partial \omega^{2r+1}} \right|_{\omega=0} = 0 \quad (r = 0, 1, \dots, L-1). \quad (15)$$

By using Eq.(12), Eq.(15) can be reduced to

$$\left. \frac{\partial^{2r+1} N(\omega)}{\partial \omega^{2r+1}} \right|_{\omega=0} = 0 \quad (r = 0, 1, \dots, L-1). \quad (16)$$

By substituting $N(\omega)$ in Eq.(13) into Eq.(16), we can derive a set of linear equations as follows

$$\sum_{n=0}^N (\tau_0 - n)^{2r+1} h(n) = 0 \quad (r = 0, 1, \dots, L-1). \quad (17)$$

It is clear that there are L equations in Eq.(17) with respect to $(N+1)$ unknown coefficients $h(n)$.

In addition to the condition given in Eq.(17), $H(z)$ is required to satisfy the condition of orthonormality and to have

the maximum number of vanishing moments, respectively. We firstly rewrite the condition of orthonormality in Eq.(9) as

$$\sum_{k=0}^{N-2n} h(2n+k)h(k) = \delta(n) = \begin{cases} 1 & (n=0) \\ 0 & (n>0) \end{cases}, \quad (18)$$

where there exist $(N+1)/2$ equations with respect to $h(n)$.

Next, $H(z)$ must have K zeros at $z = -1$ to get the specified number of vanishing moments,

$$H(z) = Q(z)(1+z^{-1})^K. \quad (19)$$

Therefore, we have

$$\left. \frac{\partial^r H(e^{j\omega})}{\partial \omega^r} \right|_{\omega=\pi} = 0 \quad (r = 0, 1, \dots, K-1), \quad (20)$$

where there are totally K equations with respect to $h(n)$. If $K+L = (N+1)/2$, then there are $K+L+(N+1)/2 = N+1$ equations in Eqs.(17), (18) and (20) with respect to $(N+1)$ unknown filter coefficients. Therefore, the filter coefficients $h(n)$ can be obtained by solving Eqs.(17), (18) and (20).

It is seen that Eq.(18) is a set of quadratic constraints on the filter coefficients $h(n)$, which is difficult to solve particularly when the filter is of higher degree. Therefore, an alternative method by linearizing the quadratic constraints into an iterative process will be introduced in the following section.

IV. AN ITERATIVE PROCEDURE

In this section, we firstly linearize the non-linear problem in Eq.(18), and then use an iterative procedure to obtain a set of scaling lowpass coefficients $h(n)$.

Let $h^{(i)}(n)$ be the filter coefficients at i^{th} iteration, and is given by

$$h^{(i)}(n) = h^{(i-1)}(n) + \Delta h^{(i)}(n). \quad (21)$$

Therefore, Eq.(18) becomes

$$\sum_{k=0}^{N-2n} [h^{(i-1)}(k+2n)h^{(i-1)}(k) + h^{(i-1)}(k+2n)\Delta h^{(i)}(k) + h^{(i-1)}(k)\Delta h^{(i)}(k+2n) + \Delta h^{(i)}(k)\Delta h^{(i)}(k+2n)] = \delta(n). \quad (22)$$

Assuming $\Delta h^{(i)}(k)$ becomes as small as i increases, then the term $\Delta h^{(i)}(k)\Delta h^{(i)}(k+2n)$ can be neglected. Thus we have

$$\begin{aligned} \sum_{k=0}^N [h^{(i-1)}(k+2n) + h^{(i-1)}(k+2n)]\Delta h^{(i)}(k) \\ = \delta(n) - \sum_{k=0}^{N-2n} h^{(i-1)}(k+2n)h^{(i-1)}(k), \end{aligned} \quad (23)$$

where $h^{(i-1)}(k) = 0$ for $k < 0$, $k > N$. Moreover, Eqs.(17) and (20) become

$$\sum_{n=0}^N (\tau_0 - n)^{2r+1} \Delta h^{(i)}(n) = \sum_{n=0}^N (n - \tau_0)^{2r+1} h^{(i-1)}(n) \quad (24)$$

$$\sum_{n=0}^N (-1)^n n^r \Delta h^{(i)}(n) = \sum_{n=0}^N (-1)^{n+1} n^r h^{(i-1)}(n). \quad (25)$$

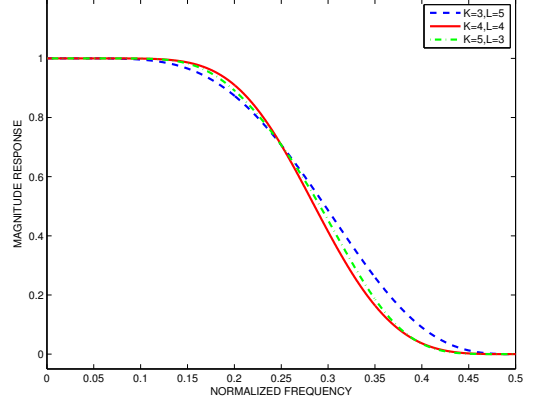


Fig. 1. Magnitude responses of $H(z)$ in Example 1.

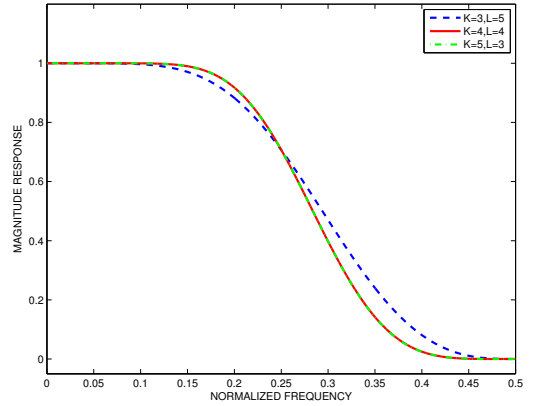


Fig. 2. Magnitude responses of $G(z)$ in Example 1.

TABLE I
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 1.

N	K	L	$E_\infty(\%)$	$E_2(\%)$
15	3	5	4.812	5.422
15	4	4	2.593	3.028
15	5	3	14.403	15.596

Therefore, we can obtain $\Delta h^{(i)}(n)$ by solving the set of linear equations in Eqs.(23), (24) and (25), if coefficients $h^{(i-1)}(n)$ are known. The filter coefficients are subsequently updated by $\Delta h^{(i)}(n)$ in Eq.(21).

To converge to the optimal solution, a set of good initial coefficients $h^{(0)}(n)$ are needed. It is known that $P(z) = H(z)H(z^{-1})$ is a linear phase half-band filter. Therefore, we firstly design $P(z)$ as the maximally flat half-band filter; The magnitude response of $H(z)$ is expressed as $|H(e^{j\omega})| = |P(e^{j\omega})|^{1/2}$. Thus, $H(e^{j\omega}) = |P(e^{j\omega})|^{1/2} e^{-j\tau_0\omega}$. Then, a set of initial coefficients $h^{(0)}(n)$ are computed by taking $(N+1)$ point IDFT.

Example 1: We have designed the scaling lowpass filters $H(z)$ and $G(z)$ with $N = 15$ by using the proposed method, where group delay $\tau_0 = 9$ was chosen. Since the number

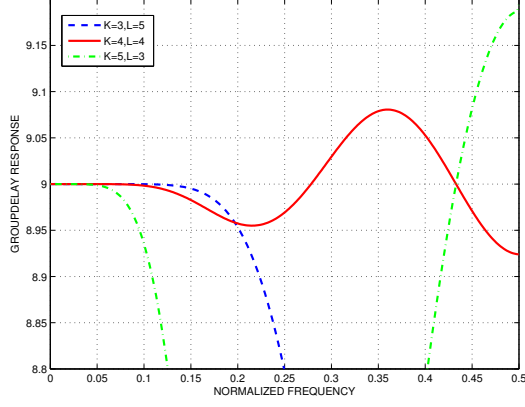


Fig. 3. Group delay responses of $H(z)$ in Example 1.

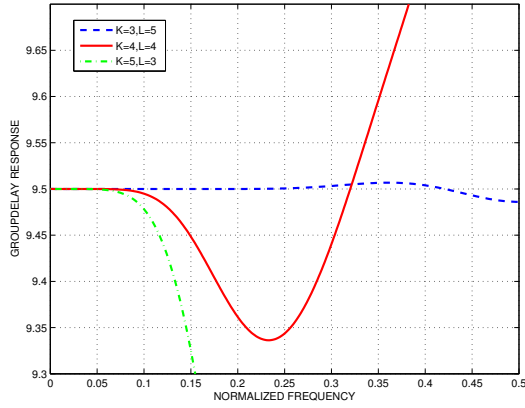


Fig. 4. Group delay responses of $G(z)$ in Example 1.

TABLE II
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 1.

K	L	τ	$E_\infty(\%)$	$E_2(\%)$
4	4	8.0	5.798	6.609
4	4	8.3	4.812	5.337
4	4	8.7	5.679	6.497
4	4	9.0	2.593	3.028
4	4	9.4	7.925	8.741

of vanishing moments can be arbitrarily specified, we choose $\{K, L\} = \{3, 5\}, \{4, 4\}, \{5, 3\}$, respectively. Fig. 1 and Fig. 2 show the magnitude responses of the scaling lowpass filters $H(z)$, $G(z)$ with different K , respectively. With an increasing K , the magnitude responses of lowpass filters become sharper. The group delay responses are shown in Fig. 3 and Fig. 4, respectively, where the group delay responses satisfy the half-sample delay condition and become more flat as L increases. The magnitude responses of $E(\omega)$ are shown in Fig. 5. It is seen in Fig. 5 that $E(\omega)$ with $\{K = 4, L = 4\}$ has the minimum error, while the filter of $\{K = 5, L = 3\}$ has

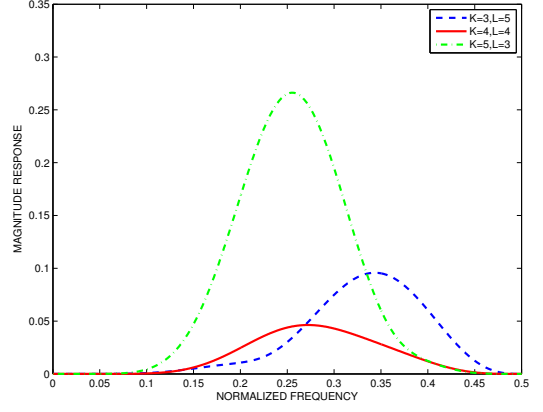


Fig. 5. Magnitude responses of $E(\omega)$ in Example 1.

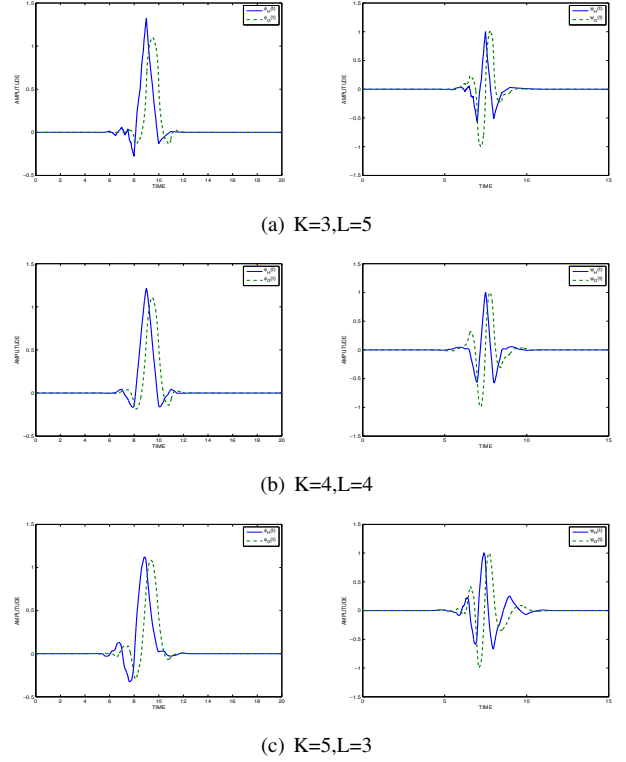


Fig. 6. Scaling and wavelet functions $\phi_H(t)$, $\phi_G(t)$, $\psi_H(t)$, $\psi_G(t)$ in Example 1.

the maximum error. Moreover, the scaling functions $\phi_H(t)$, $\phi_G(t)$ and wavelet functions $\psi_H(t)$, $\psi_G(t)$ are shown in Fig. 6, respectively. Furthermore, the complex wavelet $\Psi_c(\omega)$ are shown in Fig. 7. In Fig. 7, it is obvious that a certain degree of flatness of group delay and number of vanishing moments $\{K = 4, L = 4\}$ can achieve a better analyticity of complex wavelet. Furthermore, the analyticity measures of E_∞ and E_2 are summarized in Table I. In addition, we investigated the error function $E(\omega)$ with different τ_0 . Since the center of symmetry can be arbitrarily specified, we picked $\tau_0 = \{8.0, 8.4, 8.7, 9.0, 9.4\}$, respectively. $E(\omega)$ are presented

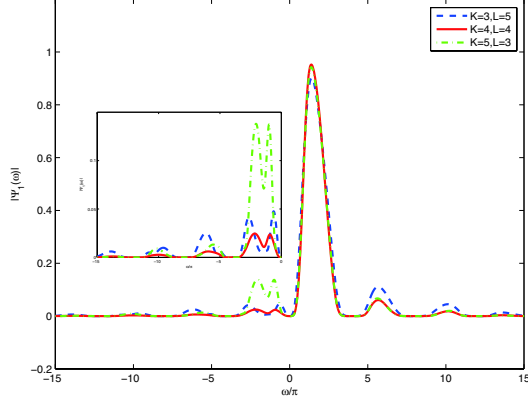


Fig. 7. Magnitude responses of $\Psi_c(\omega)$ in Example 1.

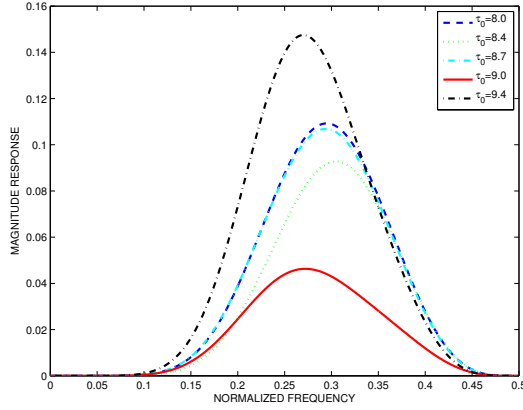


Fig. 8. Magnitude responses of $E(\omega)$ in Example 1.

in Fig. 8, while Table II summarizes the values of E_∞ and E_2 . It is obvious in Fig. 8 and Table II that the error $E(\omega)$ and analyticity measures vary with the group delay τ_0 . Therefore, a new procedure for designing $H(z)$, $G(z)$ simultaneously will be proposed in the following to improve the analyticity.

V. AN PROCEDURE FOR IMPROVED ANALYTICITY

In this section, we want to minimize the error function $E(\omega)$ between $H(z)$ and $G(z)$ to improve the analyticity.

We consider the case of $L+K < (N+1)/2$. The remaining degree of freedom is $J = (N+1)/2 - K - L$. Let $\omega_k (0 < \omega_0 < \omega_1 < \dots < \omega_{J-1})$ be the frequencies points at that $E(e^{j\omega_k}) = 0$. Therefore,

$$E(e^{j\omega_k}) = G(e^{j\omega_k}) - H(e^{j\omega_k})e^{-j\frac{\omega_k}{2}} = 0. \quad (26)$$

By substituting $\Delta h^{(i)}(n)$, $\Delta g^{(i)}(n)$ into Eq.(26), we derive a

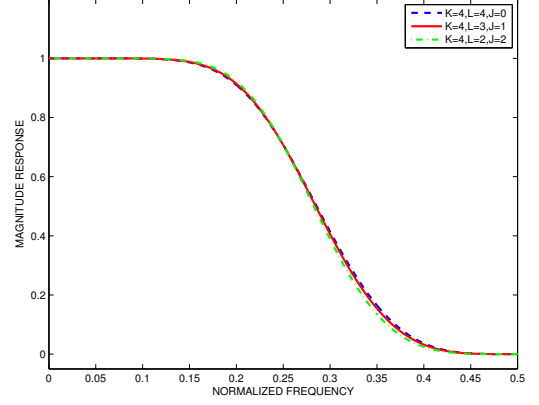


Fig. 9. Magnitude responses of $H(z)$ in Example 2.

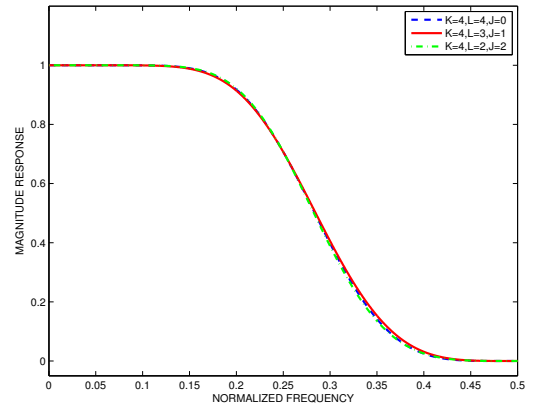


Fig. 10. Magnitude responses of $G(z)$ in Example 2.

set of linear equations as follows;

$$\begin{aligned} & \sum_{n=0}^N \Delta h^{(i)}(n) e^{-j(n+\frac{1}{2})\omega_k} - \sum_{n=0}^N \Delta g^{(i)}(n) e^{-jn\omega_k} \\ &= \sum_{n=0}^N g^{(i-1)}(n) e^{-jn\omega_k} - \sum_{n=0}^N h^{(i-1)}(n) e^{-j(n+\frac{1}{2})\omega_k}. \end{aligned} \quad (27)$$

It should be noted that since $h(n)$, $g(n)$ are real coefficients, Eq.(27) needs to be separated into the real and imaginary part, respectively. Therefore, we obtain $\Delta h^{(i)}(n)$, $\Delta g^{(i)}(n)$ together by solving a set of linear equations in three conditions of both $H(z)$ and $G(z)$, combined with Eq.(27). We use the filter coefficients of lowpass filters with the maximum degree of flatness as a set of initial coefficients $h^{(0)}(n)$, $g^{(0)}(n)$. Thus, $h(n)$ and $g(n)$ can be obtained simultaneously by the iterative procedure.

Example 2: We have designed the scaling lowpass filters $H(z)$, $G(z)$ of degree $N = 15$ with $K = 4$, $L = 3$ and $\tau_0 = 9$. Therefore, the remaining degree of freedom is $J = (N+1)/2 - K - L = 1$. We set $\omega_k = 0.544\pi$ to minimize the error function $E(\omega)$. The magnitude responses of $H(z)$ and $G(z)$ are shown in Fig. 9 and Fig. 10, respectively.

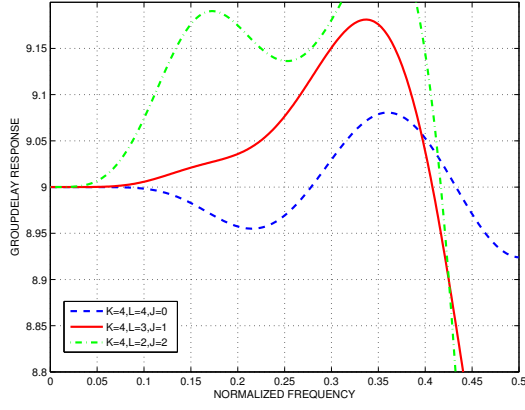


Fig. 11. Group delay responses of $H(z)$ in Example 2.

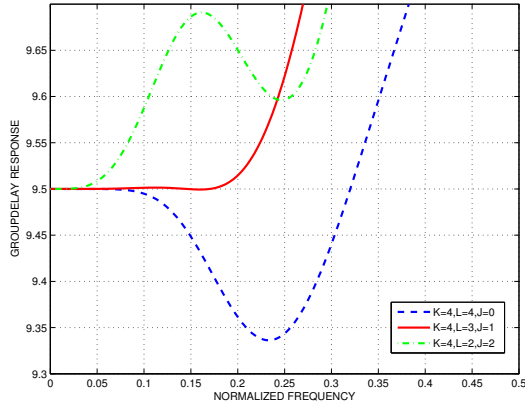


Fig. 12. Group delay responses of $G(z)$ in Example 2.

TABLE III
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 2.

N	K	L	J	$E_\infty(\%)$	$E_2(\%)$
15	4	4	0	2.593	3.028
15	4	3	1	1.006	1.293
15	4	2	2	0.547	0.598

The corresponding group delay responses are shown in Fig. 11 and Fig. 12, respectively. For comparison, the magnitude responses of $H(z)$, $G(z)$ using different degree of flatness $L = 4$, $L = 2$ with $J = 0$ and $J = 2$ are shown in Fig. 9 and Fig. 10 also. When $J = 2$, two frequency points are chosen as $\omega_k = \{0.449\pi, 0.640\pi\}$. These scaling lowpass filters have the maximally number of vanishing moments ($K = 4$). However, $E(\omega)$ are different, as shown in Fig. 13. When $J = 2$, the maximum error of $E(\omega)$ is minimum, while it is maximum when $J = 0$. It is clear that the maximum error of $E(\omega)$ can be effectively minimized at the expense of the reduced degree of flatness. In addition, the scaling functions $\phi_H(t)$, $\phi_G(t)$ and wavelet functions $\psi_H(t)$, $\psi_G(t)$ are shown in Fig. 14 respectively. Moreover the spectrum $\Psi_c(\omega)$ are

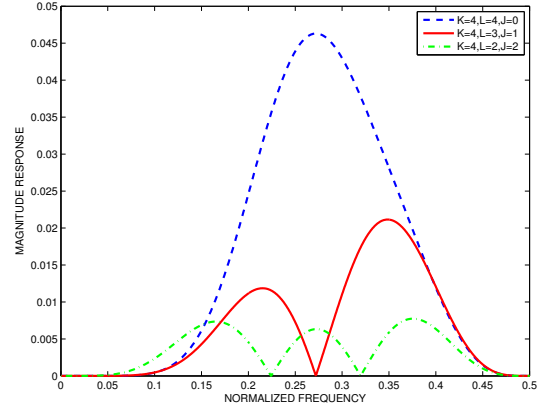


Fig. 13. Magnitude responses of $E(\omega)$ in Example 2.

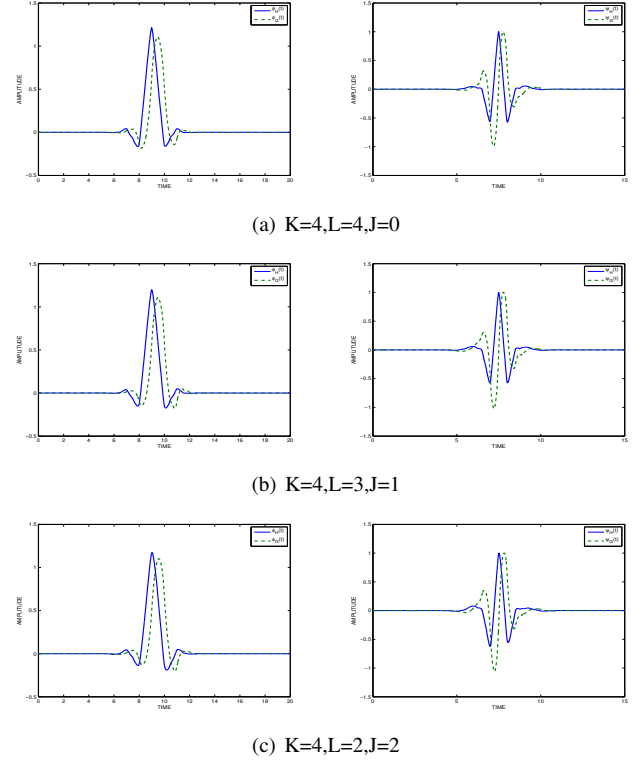


Fig. 14. Scaling and wavelet functions $\phi_H(t)$, $\phi_G(t)$, $\psi_H(t)$, $\psi_G(t)$ in Example 2.

given in Fig. 15, where the complex wavelets by our proposed method have an improved analyticity compared with the case of the maximum degree of flatness. Table III summarizes the analyticity measures of E_∞ and E_2 .

Example 3: We have chosen $\tau_0 = 11.7$ and designed the corresponding scaling lowpass filters with $N = 19$, $K = 5$ with $L = \{3, 4, 5\}$. Therefore, the remaining degree of freedom should be $J = \{2, 1, 0\}$. We set $\omega_k = 0.414\pi$ when $J = 1$ and $\omega_k = \{0.442\pi, 0.64\pi\}$ for $J = 2$, respectively. The corresponding magnitude responses of scaling lowpass filters $H(z)$, $G(z)$ are shown in Fig. 16 and Fig. 17, respectively. Since $H(z)$ and $G(z)$ have the same number of vanishing

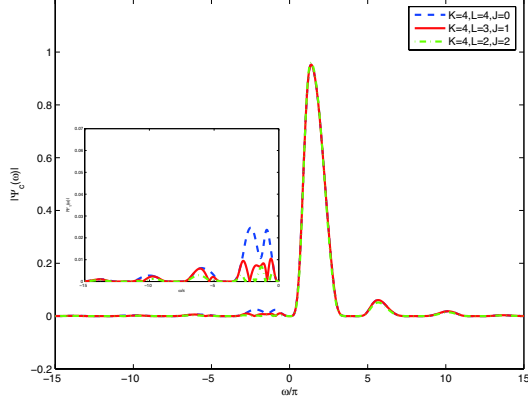


Fig. 15. Magnitude responses of $\Psi_c(\omega)$ in Example 2.

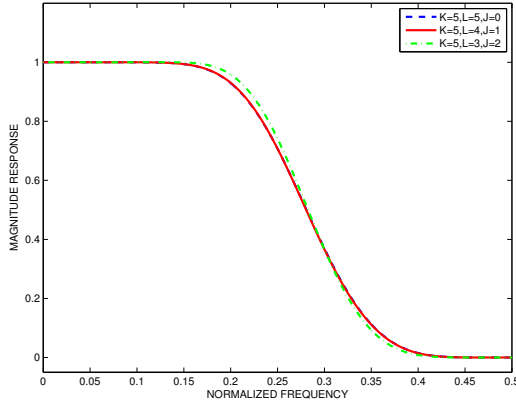


Fig. 16. Magnitude responses $H(z)$ in Example 3.

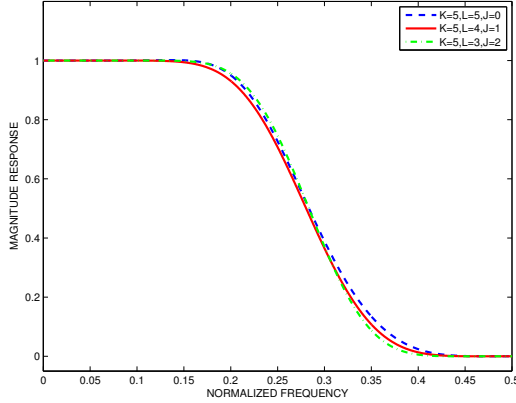


Fig. 17. Magnitude responses of $G(z)$ in Example 3.

moments, the magnitude responses in Fig. 16 and Fig. 17 are almost same. In Fig. 18 and Fig. 19, the group delay responses of the filters are given. In addition, $E(\omega)$ are shown in Fig. 20. It is obvious in Fig. 20 that with an increasing J , the error has been substantially reduced. In addition, the scaling function

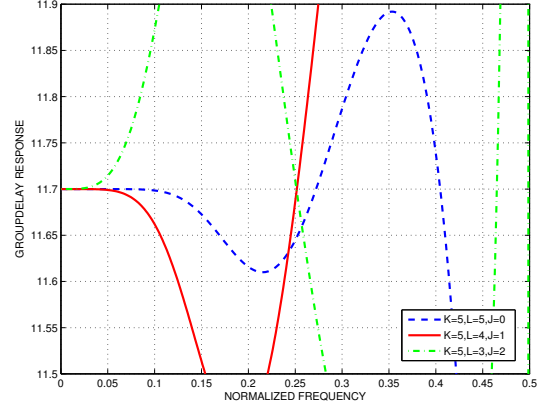


Fig. 18. Group delay responses of $H(z)$ in Example 3.

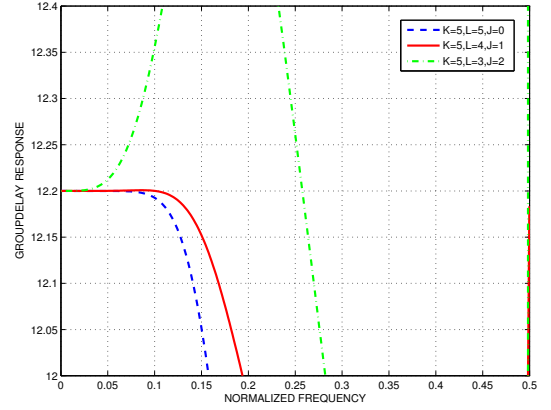


Fig. 19. Group delay responses of $G(z)$ in Example 3.

TABLE IV
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 3.

N	K	L	J	E_∞ (%)	E_2 (%)
19	5	5	0	15.647	17.115
19	5	4	1	4.572	4.748
19	5	3	2	0.965	0.809

$\phi_H(t)$, $\phi_G(t)$ and wavelet functions $\psi_H(t)$, $\psi_G(t)$ are shown in Fig. 21 respectively. Moreover, $\Psi_c(\omega)$ are given in Fig. 22, where the analyticity of complex wavelet has been greatly improved. Finally, we evaluated the analyticity measures, E_∞ and E_2 and summarized in Table IV.

VI. CONCLUSION

In this paper, we have proposed a new method for designing a class of Hilbert pairs of almost symmetric orthogonal wavelets with arbitrary center of symmetry. We have firstly designed two scaling lowpass filters independently. Next, we have used the remaining degree of freedom to minimize the error function $E(\omega)$ between two scaling lowpass filters. Therefore, two lowpass filters can be designed simultaneously

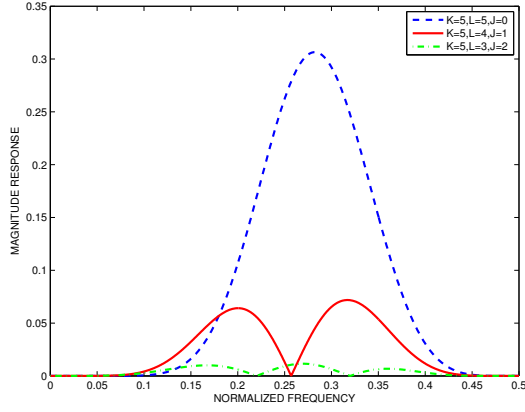
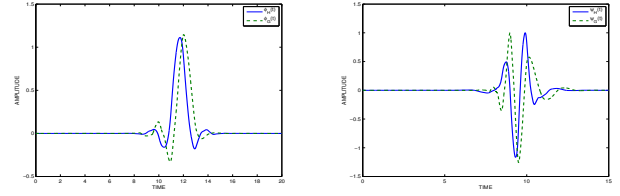


Fig. 20. Magnitude responses of $E(\omega)$ in Example 3.

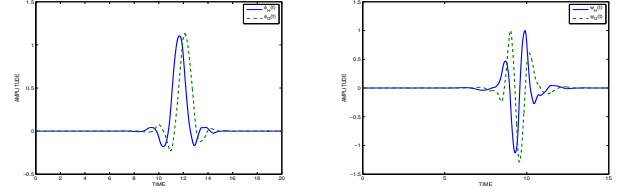
by linearizing the condition of orthogonality of wavelets. The proposed design method is computationally efficient. Moreover, the filter coefficients can be computed easily by iteratively solving a set of linear equations only. As a result, the obtained orthogonal wavelet filters have flat group delay responses and the specified number of vanishing moments, while minimizing the error to improve the analyticity. Finally, several examples are presented to demonstrate the effectiveness of the design method proposed in this paper.

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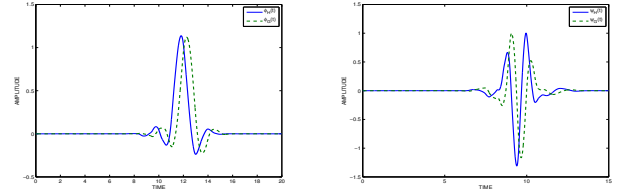
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(a) $K=5, L=5, J=0$



(b) $K=5, L=4, J=1$



(c) $K=5, L=3, J=2$

Fig. 21. Scaling and wavelet functions $\phi_H(t)$, $\phi_G(t)$, $\psi_H(t)$, $\psi_G(t)$ in Example 3.

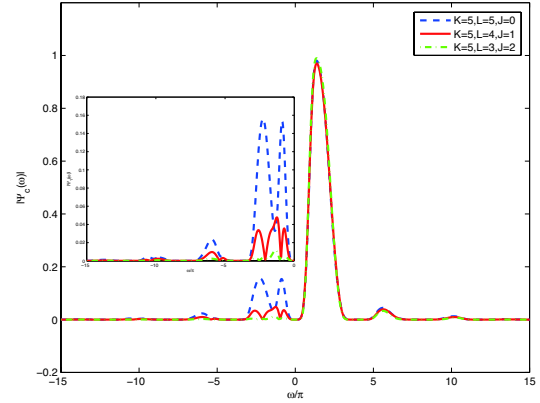


Fig. 22. Magnitude responses of $\Psi_c(\omega)$ in Example 3.

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