Design of *M*th-band FIR Linear Phase Filters

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***Abstract*—This paper proposes an improved method for the Chebyshev approximation of *M*th-band FIR linear phase filters using the Remez exchange algorithm. It is known from the time-domain condition that the magnitude responses of *M*th-band FIR filters are dependent on each other in passband and stopband. In the conventional method, the Remez exchange algorithm was applied only in stopband to minimize the stopband error [8]. In this paper, the Remez exchange algorithm is applied in passband and a part of stopband to improve the magnitude response. In the proposed method, the filter coefficients are computed only by solving a set of linear equations, and the optimal coefficients can be easily obtained through a few iterations. The proposed method is more computationally efficient than the existing methods.**

***Index Terms*—Chebyshev approximation; *M*th-band filter; FIR filter; linear phase filter**

I. Introduction

*M*th-band FIR filters have been found numerous applications in filter banks, nonuniform sampling, and so on [1], [2]. *M*th-band FIR filter is required to be lowpass, and its impulse response is exactly zero-crossing. Many design methods have been proposed for designing *M*th-band FIR filter in [1]∼[10]. Most of the existing methods have used linear programming techniques and/or nonlinear optimization methods, and are generally time-consuming [1]∼[6]. It is known that the magnitude responses of *M*th-band FIR filters are dependent on each other in passband and stopband [8], [9], thus the Remez exchange algorithm cannot be applied in the full passband and stopband. In [8], we have proposed a computationally efficient design method using the Remez exchange algorithm only in stopband to minimize the stopband error. However, the magnitude response in passband was not taken into account in this design method. Sometimes the resulting passband error is larger than the stopband one.

In this paper, we discuss the Chebyshev approximation of *M*th-band FIR linear phase filters, where the maximum magnitude error is minimized both in passband and stopband. We review the conventional design method in [8], and then propose an improved method of *M*th-band FIR filters. To minimize the magnitude error both in passband and stopband, we apply the Remez exchange algorithm in passband and a part of stopband to formulate the filter design problem. Therefore, the filter coefficients are computed only by solving a set of linear equations, and the optimal coefficients can be easily obtained through a few iterations. The proposed design method is computationally efficient. Finally, two design examples are presented to demonstrate the effectiveness of the proposed design method.

II. *M*th-band FIR Linear Phase Filters

Let $h(n)$ be the impulse response of FIR filter. Its transfer function $H(z)$ can be given by

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n}, \tag{1}$$

where N is the filter degree. If the impulse response is symmetric, i.e., $h(n) = h(N-n)$, then FIR filter has the exactly linear phase response.

When $H(z)$ is required to be *M*th-band linear phase filter, its impulse response must satisfy the following zero-crossing condition;

$$\begin{cases} h(\frac{N}{2}) = \frac{1}{M} & \\ h(\frac{N}{2} + kM) = 0 & (k = \pm 1, \pm 2, \cdots) \end{cases}, \tag{2}$$

where N, M are integer and N is an even number.

It is known that the *M*th-band filter is required to be lowpass, and then the desired magnitude response of $H(z)$ is given by

$$|H_d(e^{j\omega})| = \begin{cases} 1 & (0 \le \omega \le \omega_p) \\ 0 & (\omega_s \le \omega \le \pi) \end{cases}, \tag{3}$$

where $\omega_p = (1-\rho)\frac{\pi}{M}$ and $\omega_s = (1+\rho)\frac{\pi}{M}$ are the cutoff frequencies of passband and stopband respectively, and ρ is a rolloff rate.

By substituting the time-domain condition in Eq.(2) into Eq.(1), we can obtain the magnitude response of $H(z)$ as follows;

$$|H(e^{j\omega})| = \frac{1}{M} + \sum_{\substack{n=1 \\ \neq kM}}^{N/2} a_n \cos(n\omega), \tag{4}$$

where $a_n = 2h(N/2 \pm n)$. Therefore, the design problem of *M*th-band FIR linear phase filters can be reduced to the approximation of the magnitude response in Eq.(4) to the desired magnitude response in Eq.(3).

In this paper, we consider the Chebyshev approximation of *M*th-band FIR linear phase filters. First, we define the weighted error function of the magnitude response as follows;

$$E(\omega) = W(\omega)\{|H(e^{j\omega})| - |H_d(e^{j\omega})|\}, \tag{5}$$

where $W(\omega)$ is a real and positive weighting function. Therefore, the Chebyshev approximation problem consists in finding the filter coefficients a_n that will minimize the following Chebyshev norm;

$$||E(\omega)|| = \max_{\omega \in R} |E(\omega)|, \tag{6}$$

where R is the approximation band including both of passband and stopband.

III. *M*TH-BAND FILTER PROPERTY

Before designing $H(z)$, we firstly investigate the property of *M*th-band FIR filters. It can be seen that the magnitude response $|H(e^{j\omega})|$ in Eq.(4) always satisfies

$$\sum_{m=0}^{M-1} |H(e^{j(\omega + \frac{2m\pi}{M})})| \equiv 1 \quad , \tag{7}$$

which means that the sum of the magnitudes $|H(e^{j\omega})|$ at the frequency points $\omega = \omega_0 + \frac{2m\pi}{M}$ for $m = 0, 1, \cdots, M-1$ keep unity regardless of what the filter coefficients a_n are. It is because the time-domain condition in Eq.(2) has been included in Eq.(4).

Since $|H(e^{j(2\pi-\omega)}) = |H(e^{j\omega})|$, we have

$$|H(e^{j\omega_0})| = 1 - \sum_{m=1}^{M-1} |H(e^{j\omega_m})|, \tag{8}$$

where $\omega_m = \frac{2m\pi}{M} + \omega_0$ for $1 \le m \le \lfloor \frac{M-1}{2} \rfloor$, $\omega_m = \frac{2(M-m)\pi}{M} - \omega_0$ for $\lfloor \frac{M+1}{2} \rfloor \le m \le M-1$, and $\lfloor x \rfloor$ denotes the integer part of x. Assume that ω_0 is in passband, i.e., $0 \le \omega_0 \le \omega_p$, then ω_m should be in stopband, i.e., $\omega_s \le \omega_m \le \pi$ for $1 \le m \le M-1$. It is clear in Eq.(8) that if its stopband response is 0, then the magnitude response of $H(z)$ becomes 1 in passband. Therefore, the magnitude error in passband is dependent on the stopband error. Let δ_s be the maximum magnitude error in stopband, the maximum magnitude error δ_p in passband is given by

$$\delta_p \le (M-1)\delta_s. \tag{9}$$

Note that $\delta_p = (M-1)\delta_s$ is the worst case, and δ_p is usually much smaller than it in the practical design.

In [8] and [9], the filter design was concentrated on shaping the stopband response. That is, the maximum magnitude error was minimized only in stopband;

$$\min ||E(\omega)|| = \min\{\max_{\omega_s \le \omega \le \pi} |E(\omega)|\}. \tag{10}$$

It can be explained according to the zero location. There are $I = \frac{N}{2} - \lfloor \frac{N}{2M} \rfloor)$ unknown coefficients a_n in Eq.(4), thus $H(z)$ has $2I$ independent zeros. In [8], these independent zeros were located on the unit circle to minimize the magnitude error in stopband by using the Remez exchange algorithm. However, the passband response was not taken into account in this design method. Sometimes the resulting magnitude error in passband becomes larger than that in stopband, as shown in Fig.1.

IV. IMPROVED DESIGN OF *M*TH-BAND FIR FILTERS

In this section, we will discuss the design of *M*th-band FIR linear phase filters, where the maximum magnitude error is minimized both in passband and stopband. It is known that the Remez exchange algorithm is an efficient tool for designing FIR linear phase filters, but cannot be directly used to design *M*th-band FIR filters, because of the time-domain condition in Eq.(2). In [8], we had proposed a design method of *M*th-band FIR linear phase filters by applying the Remez exchange algorithm in stopband. In the following, we firstly review the conventional method proposed in [8], and then propose an improved method for the Chebyshev approximation of Mth-band FIR linear phase filters.

A. The Conventional Method

It is known in [8] that *M*th-band FIR linear phase filters have $2I$ independent zeros, and these independent zeros have been located on the unit circle to minimize the magnitude error in stopband. When all $2I$ zeros are located on the unit circle, there exist $(I+1)$ extremal frequencies in stopband $[\omega_s, \pi]$. Thus, $(I+1)$ extremal frequencies ω_i $(i = 0, 1, \cdots, I)$ can be selected in stopband, as shown in Fig.2;

$$\omega_s = \omega_0 < \omega_1 < \cdots < \omega_I = \pi. \tag{11}$$

By using the Remez exchange algorithm, we formulate the error function in Eq.(5) in such a way that its amplitude is equal, and its sign alternates at the extremal frequencies ω_i;

$$E(\omega_i) = (-1)^i \, \delta, \tag{12}$$

where $\delta (> 0)$ is the magnitude error.

Substituting Eqs.(3), (4) and (5) into Eq.(12), we have

$$\sum_{\substack{n=1 \\ \neq kM}}^{N/2} a_n \cos(n\omega_i) - \frac{(-1)^i}{W(\omega_i)} \, \delta = -\frac{1}{M}, \tag{13}$$

which is a set of linear equations. There are $(I+1)$ unknowns (I coefficients a_n and one error δ) with respect to $(I+1)$ extremal frequencies, thus the filter coefficients a_n can be obtained by solving the linear equations in Eq.(13). By using the obtained coefficients a_n, we compute the magnitude response and search the peak frequencies $\bar{\omega}_i$ in stopband. However, the initially selected extremal frequencies ω_i cannot

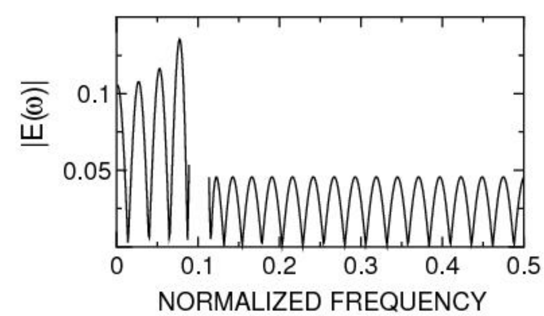


Fig. 1. The magnitude error of the conventional *M*th-band FIR filter.

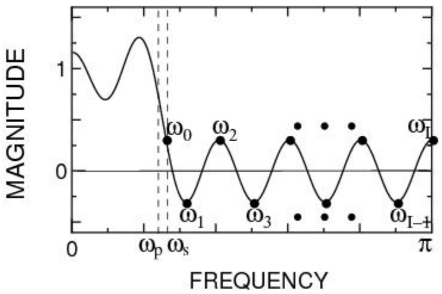


Fig. 2. The conventional selection of extremal frequencies ω_i.

be guaranteed to be the same as the peak frequencies $\bar{\omega}_i$. Therefore, we set the obtained peak frequencies $\bar{\omega}_i$ as the extremal frequencies ω_i in the next iteration, and solve the linear equations in Eq.(13) to obtain a set of coefficients a_n again. The above procedure is iterated until the extremal frequencies ω_i are consistent with the peak frequencies $\bar{\omega}_i$. If the extremal frequencies do not change, we can obtain the equiripple magnitude response in stopband. Therefore, the obtained filter coefficients are optimal in the Chebyshev sense in stopband. However, the passband error cannot be controlled directly by this method. Since the magnitude response in passband is dependent on the stopband response as shown in Eq.(8), the passband error may be smaller or larger than the stopband error. When the passband error is smaller than the stopband one, the optimal solution have been obtained. If the passband error is larger than the stopband one as shown in Fig.1, however, it cannot be said that it is optimal in both of passband and stopband.

B. The Improved Method

In the case where the passband error is larger than the stopband one, it is required to controll directly the magnitude error in passband. To minimize the magnitude error in both of passband and stopband, we propose an improved selection of extremal frequencies ω_i.

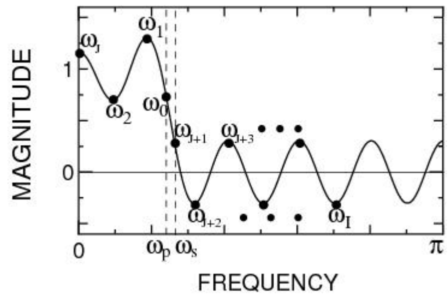
It is known that there are $\frac{N}{2}+2$ extremal frequencies for FIR linear phase filters designed by using the Remez exchange algorithm. As shown in the above-mentioned subsection, Mth-band FIR linear phase filters have $(I+1)$ extremal frequencies in stopband, thus, there are $(J+1)$ extremal frequencies in passband, where $J=\frac{N}{2}-I=\lfloor\frac{N}{2M}\rfloor$. We firstly select $(J+1)$ extremal frequencies ω_i $(i=0,1,\cdots,J)$ in passband as shown in Fig.3;

$$\omega_p=\omega_0>\omega_1>\cdots>\omega_J=0, \tag{14}$$

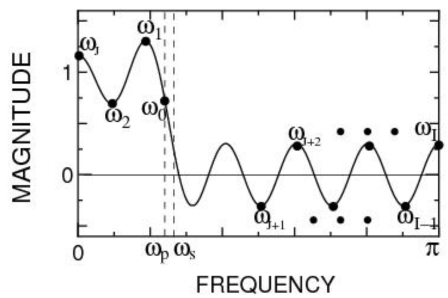
and then formulate the error function $E(\omega)$ as in Eq.(12) to minimize the passband error. Substituting Eqs.(3), (4) and (5) into Eq.(12), we have

$$\sum_{\substack{n=1\\ \neq kM}}^{N/2} a_n\cos(n\omega_i)+\frac{(-1)^i}{W(\omega_i)}\,\delta=1-\frac{1}{M}. \tag{15}$$

In addition, we need to minimize the stopband error too. There are $(I+1)$ extremal frequencies in stopband, howerer,



(a)



(b)

Fig. 3. The proposed selection of extremal frequencies ω_i in passband and stopband: (a) Method 1, (b) Method 2.

we need only $(I-J)$ extremal frequencies, since the time-domain condition in Eq.(2) exists. We must select $(I-J)$ extremal frequencies ω_i $(i=J+1,J+2,\cdots,I)$ in a part of stopband. In this paper, we propose two methods (**Method 1** and **Method 2**). **Method 1** selects the extremal frequencies ω_i from ω_s as shown in Fig.3(a);

$$\omega_s=\omega_{J+1}<\omega_{J+2}<\cdots<\omega_I<\pi, \tag{16}$$

where the peak frequencies are not selected in the neighborhood of $\omega=\pi$. On the other hand, **Method 2** selects the extremal frequencies ω_i from π as shown in Fig.3(b);

$$\omega_s<\omega_{J+1}<\omega_{J+2}<\cdots<\omega_I=\pi, \tag{17}$$

where the peak frequencies are not selected in the neighborhood of $\omega=\omega_s$. We then formulate the error function $E(\omega)$ as in Eq.(12) to minimize the error in a part of stopband. The same formula is obtained as in Eq.(13), but $i=J+1,J+2,\cdots,I$. Therefore, there are totally $(I+1)$ equations, and we can obtain a set of filter coefficients a_n by solving this system of linear equations. Moreover, the equiripple response can be obtained in a part of stopband and passband through a few iterations. It should noted that since some peak frequencies are not selected in stobband, the error maybe become larger in the band where the peak frequencies are not selected. Therefore, we need to choose one better solution from two solutions obtained by **Method 1** and **Method 2**, that is, the filter having the minimum error in both of passband and stopband is chosen. The design algorithm is shown in the following in detail.

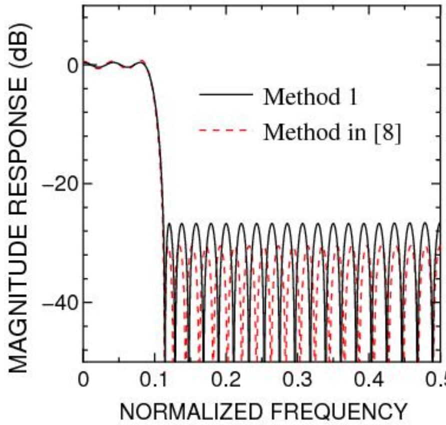


Fig. 4. Magnitude responses of *M*th-band FIR filters in Example 1.

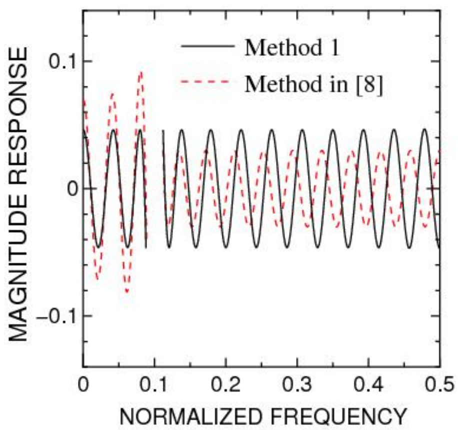


Fig. 5. Magnitude errors of *M*th-band FIR filters in Example 1.

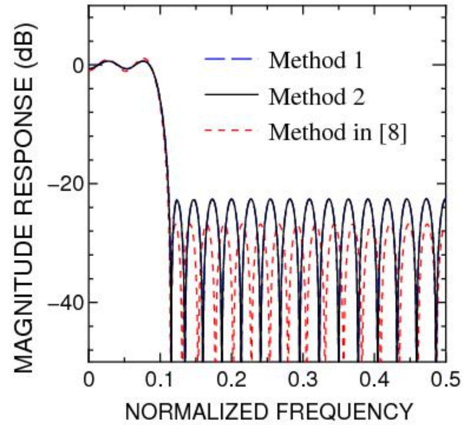


Fig. 6. Magnitude responses of *M*th-band FIR filters in Example 2.

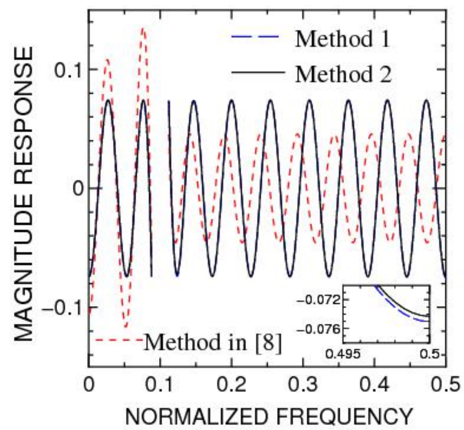


Fig. 7. Magnitude errors of *M*th-band FIR filters in Example 2.

C. Design Algorithm

1. Read specification of *M*th filter N, M, ρ and $W(\omega)$.
2. Select $(I+1)$ initial extremal frequencies ω_i equally spaced in passband and a part of stopband as shown in Fig.3.
3. Solve the linear equations in Eq.(13) and Eq.(15) to obtain a set of filter coefficients a_n.
4. Compute the magnitude response using the obtained coefficients a_n, and search the peak frequencies $\bar{\omega}_i$ in passband and stopband.
5. If $|\bar{\omega}_i - \omega_i| < \epsilon$ $(i = 0, 1, \cdots, I)$ are satisfied, then stop, else go to **6**, where ϵ is a prescribed small constant, e.g., $\epsilon = 10^{-10}$.
6. set $\omega_i = \bar{\omega}_i$ $(i = 0, 1, \cdots, I)$, then go to **3**.

V. Design Examples

In this section, we present two design examples to demonstrate the effectiveness of the proposed method for designing *M*th-band FIR linear phase filters.

Example 1 We consider the design of *M*th-band FIR linear phase filter with $N = 48, M = 5, \rho = 0.12, W(\omega) = 1$. Firstly, we have designed the filter by using the conventional method proposed in [8]. The resulting magnitude response and error are shown in the dotted line in Fig.4 and Fig.5, respectively. It is seen in Fig.5 that the equiripple response is obtained in stopband, however, the passband error is larger than the stopband one. To improve the passband magnitude response, we have used **Method 1** to design this filter, and its magnitude response and error are also shown in the solid line in Fig.4 and Fig.5. It is clear that the maximum magnitude error is smaller than the conventional method in [8].

Example 2 We consider the design of *M*th-band FIR linear phase filter with $N = 38, M = 5, \rho = 0.12, W(\omega) = 1$. We have used the conventional method in [8], **Method 1** and **Method 2** proposed in this paper to design this filter. The resulting magnitude responses and errors are shown in Fig.6 and Fig.7, respectively. It is seen in Fig.7 that the maximum magnitude error of the filter designed by the conventional method is larger than that of the proposed methods in passband. The magnitude responses obtained by **Method 1** and **Method 2** are almost the same, but slightly differ nearby $\omega = \pi$, where **Method 2** has the minimum error. Thus the filter designed by using **Method 2** is the best.

VI. CONCLUSION

In this paper, we have discussed the Chebyshev approximation of *M*th-band FIR linear phase filters, where the maximum magnitude error is minimized both in passband and stopband. We have proposed an improved design method for the Chebyshev approximation of *M*th-band FIR filters. To minimize the magnitude error in both of passband and stopband, we have applied the Remez exchange algorithm in passband and a part of stopband to formulate the design problem of *M*th-band FIR linear phase filters. Therefore, the filter coefficients are computed only by solving a set of linear equations, and the optimal coefficients can be easily obtained through a few iterations. The proposed design method is more computationally efficient than the existing design methods. Finally, two design examples have been presented to demonstrate the effectiveness of the design method proposed in this paper.

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