PAPER

A New Class of Hilbert Pairs of Almost Symmetric Orthogonal Wavelet Bases

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SUMMARY This paper proposes a new class of Hilbert pairs of almost symmetric orthogonal wavelet bases. For two wavelet bases to form a Hilbert pair, the corresponding scaling lowpass filters are required to satisfy the half-sample delay condition. In this paper, we design simultaneously two scaling lowpass filters with the arbitrarily specified flat group delay responses at $\omega = 0$, which satisfy the half-sample delay condition. In addition to specifying the number of vanishing moments, we apply the Remez exchange algorithm to minimize the difference of frequency responses between two scaling lowpass filters, in order to improve the analyticity of complex wavelets. The equiripple behavior of the error function can be obtained through a few iterations. Therefore, the resulting complex wavelets are orthogonal and almost symmetric, and have the improved analyticity. Finally, some examples are presented to demonstrate the effectiveness of the proposed design method.

key words: *DTCWT, Hilbert transform pair, almost symmetric orthogonal wavelets, FIR filter, Remez exchange algorithm*

1. Introduction

The Dual-Tree Complex Wavelet Transform (DT$\mathcal{C}$WT) was originally introduced by Kingsbury [6], and has been found to be successful in many applications of signal and image processing [6]–[16]. The DT$\mathcal{C}$WT can provide the significant improvement over the conventional discrete wavelet transform (DWT) proposed in [1], e.g., it is of approximate shift invariance, has an enhanced directional selectivity for multidimensional signals, and gives the explicit phase information [14]. The DT$\mathcal{C}$WT is generally constructed by an approximate Hilbert transform pair of wavelets. It has been shown in [11], [13]–[17] that the necessary and sufficient condition for two wavelet bases to be a pair of Hilbert transform is that the two corresponding lowpass filters must satisfy the half-sample delay condition.

Several design procedures for constructing Hilbert pairs of orthogonal wavelets had been proposed in [6]–[24]. In [11], Selesnick had proposed the common-factor design technique based on the maximally-flat allpass filter. This method was simple and effective, because the approximation accuracy of the half-sample delay is controlled only by the allpass filter. The common-factor method had been generalized by using IIR filters with numerator and denominator of different degree to obtain a new class of IIR orthogonal solutions in [23]. However, the wavelet filters obtained by the common-factor method have non-linear phase responses, resulting in asymmetric wavelet bases. To obtain symmetric wavelet bases, the Q-shift filter was proposed by Kingsbury in [8], [9], [12]. In [8], two scaling lowpass filters were selected to be the time-reversed version of each other. Therefore, the group delay of lowpass filter is required to be 1/4 (quarter) or 3/4-sample from the half-sample delay condition, and thus the filter was called Q-shift filter. Some design methods for Q-shift filters had been proposed in [9], [12], [19], [21] to improve the vanishing moments, symmetry and so on. In addition, SSH (symmetric self-Hilbertian) filter had been proposed by Tay in [17] and its design had been discussed in [18], [20], [22]. In principle, the SSH filter is the same as the Q-shift filter and then must have a group delay of 1/4-sample. Moreover, a class of almost symmetric orthogonal Hilbert pair of wavelets had been also proposed in [24], where AOS (Almost-Odd-Symmetric)/AES (Almost-Even-Symmetric) filters were designed by approximating the symmetric impulse responses instead of group delay. However, the group delay is fixed to whole-sample or half-sample, since the center of symmetry is set to half the filter degree. The purpose of this paper is to propose a design method for the Hilbert pairs of wavelets with the group delay arbitrarily specified by the user, including 1/4, half and whole-sample delay.

In many applications of signal processing, digital filters with the specified (fractional or integer) group delay are often needed [3]. For the conventional DWTs, nearly symmetric orthogonal wavelets, e.g., coiflets, had been proposed in [1, chapter 8.2], and the original coiflets had been also generalized by varying the group delay at $\omega = 0$, where non-integer group delay was used to obtain a rich class of new wavelets [4], [5]. The DT$\mathcal{C}$WT was proposed to improve the drawbacks of DWT, e.g., lack of shift invariance. Similarly, we can obtain a rich class of new DT$\mathcal{C}$WTs by varying the group delay.

In this paper, we propose a design method for a new class of Hilbert pairs of almost symmetric orthogonal wavelets. We specify the degree of flatness of group delay at $\omega = 0$ and the number of vanishing moments, and then apply the Remez exchange algorithm to minimize the difference between two scaling lowpass filters in the frequency domain, in order to improve the analyticity of complex wavelets. Moreover, two scaling lowpass filters are obtained simultaneously by iteratively solving a set of equations. Therefore, the optimal solution is attained through a few iterations. As a result,

Manuscript received October 14, 2015.
Manuscript revised December 25, 2015.

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DOI: 10.1587/transfun.E99.A.884

the complex wavelets are orthogonal and almost symmetric, and have the improved analyticity. Differently from the Q-shift and AOS/AES filters, the group delay of the filter can be arbitrarily specified by the user. Finally, some examples are presented to demonstrate the effectiveness of the proposed method.

This paper is organized as follows. Section 2 briefly reviews Hilbert transform pair of symmetric orthogonal wavelets and the half-sample delay condition. In Sect. 3, the design method of almost symmetric orthogonal wavelet filters with specified group delay responses is proposed. A new design procedure for improving the analyticity of complex wavelets is given by using the Remez exchange algorithm in Sect. 4. In Sect. 5, the proposed filter banks are applied to signal denoising to prove the effectiveness. Finally, Sect. 6 contains a conclusion.

2. Hilbert Pair of Symmetric Orthogonal Wavelets

It is known in [6] that the DT$\mathcal{C}$WT employs two real DWTs, where the first DWT generates the real part of DT$\mathcal{C}$WT and the other one is its imaginary part.

Let $\{\phi_i(t), \psi_i(t)\}$ be the scaling and wavelet functions of two DWTs, where $i = 1, 2$. It had been proven in [11], [13]–[17] that two wavelet functions $\psi_i(t)$ are a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \tag{1}$$

that is,

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases} \tag{2}$$

if and only if the corresponding scaling lowpass filters $H_1(z)$ and $H_2(z)$ satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j(2M+\frac{1}{2})\omega} \quad (|\omega| < \pi), \tag{3}$$

where $\Psi_1(\omega), \Psi_2(\omega)$ are the Fourier transform of $\psi_1(t), \psi_2(t)$, respectively, and M is an integer. Equation (3) is the generalized half-sample delay condition[†]. Specifically, the scaling lowpass filters should be offset from another one by a half sample. It is seen in Eq. (3) that $H_2(e^{j\omega})$ needs to be approximated to $H_1(e^{j\omega})e^{-j(2M+\frac{1}{2})\omega}$. Therefore, we define the error function $E(\omega)$ as

$$E(\omega) = H_2(e^{j\omega}) - H_1(e^{j\omega})e^{-j(2M+\frac{1}{2})\omega}. \tag{4}$$

If two wavelet functions are an ideal pair of Hilbert transform, the complex wavelet $\psi_c(t) = \psi_1(t) + j\psi_2(t)$ is analytic, i.e., the spectrum is one-sided;

$$\Psi_c(\omega) = \Psi_1(\omega) + j\Psi_2(\omega) = \begin{cases} 2\Psi_1(\omega) & (\omega > 0) \\ 0 & (\omega < 0) \end{cases} \tag{5}$$

[†]In this paper, $M = 0$ is used in all design examples.

which is ideally 0 in the negative frequency domain. However, it is impossible to achieve the ideal Hilbert transform with realizable filters. To evaluate the analyticity of complex wavelets, we use the p-norm of the spectrum $\Psi_c(\omega)$ to define an objective measure of quality as

$$E_p = \frac{||\Psi_c(\omega)||_{p,(-\infty,0)}}{||\Psi_c(\omega)||_{p,(0,\infty)}}, \tag{6}$$

where

$$||\Psi_c(\omega)||_{p,\Omega} = \left(\int_{\Omega} |\Psi_c(\omega)|^p d\omega\right)^{\frac{1}{p}}. \tag{7}$$

If $p = \infty$, $E_\infty = \lim_{p\to\infty} E_p$ evaluates the peak error in the negative frequency domain with respect to that in the positive frequency domain. If $p = 2$, E_2 evaluates the square root of the negative frequency energy with respect to that in the positive frequency domain. In this paper, we will use E_∞ and E_2 to evaluate the analyticity of the complex wavelets.

In two channel filter banks, the scaling lowpass filter $H_i(z)$ is used as lowpass filter in analysis filter bank, then $G_i(z) = z^{-N}H_i(-z^{-1})$ is highpass filter of analysis filter bank. In synthesis filter bank, $\tilde{H}_i(z) = z^{-N}H_i(z^{-1})$ is lowpass and $\tilde{G}_i(z) = H_i(-z)$ is highpass filter, where N is the filter degree and must be an odd number. Therefore, the scaling lowpass filter has to satisfy the condition of orthogonality to generate the orthonormal wavelet bases;

$$H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2. \tag{8}$$

Moreover, two scaling lowpass filters are required to have linear phase responses for wavelet bases to be symmetric. That is, the desired phase response $\theta_i^d(\omega)$ of $H_i(z)$ is

$$\theta_i^d(\omega) = -\tau_i\omega. \tag{9}$$

From the half-sample delay condition, $\tau_2 = \tau_1 + 2M + 1/2$ is required. Since the group delay τ_1 can be arbitrarily given, the almost symmetric scaling functions with the arbitrarily specified center of symmetry can be obtained.

3. Design of Almost Symmetric Orthogonal Wavelet Filters

In this section, we discuss the design of scaling lowpass filters $H_i(z)$ with the specified flat group delay and the specified number of vanishing moments. Many criterions can be used to approximate the group delay, e.g., the maximally flat, the weighted least square, the equiripple approximation, and so on. To obtain a number of vanishing moments on scaling functions [1], [5], we consider the flatness condition of group delay.

The transfer functions $H_i(z)$ of FIR filters are given by

$$H_i(z) = \sum_{n=0}^{N} h_i(n)z^{-n}, \tag{10}$$

where $h_i(n)$ are real filter coefficients. Let $\theta_i(\omega)$ be the phase response of $H_i(z)$, the difference $\theta_i^e(\omega)$ between $\theta_i(\omega)$ and $\theta_i^d(\omega)$ is given by

$$\theta_i^e(\omega) = \theta_i(\omega) - \theta_i^d(\omega) = \tan^{-1}\frac{N_i(\omega)}{D_i(\omega)}, \tag{11}$$

where

$$\begin{cases} N_i(\omega) = \displaystyle\sum_{n=0}^{N} h_i(n)\sin\{(\tau_i - n)\omega\} \\ D_i(\omega) = \displaystyle\sum_{n=0}^{N} h_i(n)\cos\{(\tau_i - n)\omega\} \end{cases} \tag{12}$$

The group delay response $\tau_i(\omega)$ is required to be flat with the specified degree of flatness at $\omega = 0$;

$$\begin{cases} \tau_i(0) = \tau_i \\ \left.\dfrac{\partial^{2r}\tau_i(\omega)}{\partial\omega^{2r}}\right|_{\omega=0} = 0 \quad (r = 1, 2, \cdots, L-1) \end{cases} \tag{13}$$

where L (> 0) is a parameter that controls the degree of flatness. Since $\tau_i(\omega) = -\frac{\partial\theta_i(\omega)}{\partial\omega}$, Eq. (13) is equivalent to

$$\left.\frac{\partial^{2r+1}\theta_i^e(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0 \quad (r = 0, 1, \cdots, L-1). \tag{14}$$

By using Eq. (11), Eq. (14) can be reduced to

$$\left.\frac{\partial^{2r+1}N_i(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0 \quad (r = 0, 1, \cdots, L-1). \tag{15}$$

We substitute $N_i(\omega)$ in Eq. (12) into Eq. (15), and then derive a set of linear equations;

$$\sum_{n=0}^{N}(\tau_i - n)^{2r+1}h_i(n) = 0 \quad (r = 0, 1, \cdots, L-1). \tag{16}$$

It is clear that there are L equations in Eq. (16) with respect to $(N+1)$ unknown coefficients $h_i(n)$.

In addition to the phase condition, the wavelets are also required to have the specified number of vanishing moments and satisfy the condition of orthonormality. From the viewpoint of regularity, $H_i(z)$ must have K zeros at $z = -1$;

$$H_i(z) = Q(z)(1 + z^{-1})^K, \tag{17}$$

which is equivalent to

$$\left.\frac{\partial^r H_i(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\pi} = 0 \quad (r = 0, 1, \cdots, K-1). \tag{18}$$

By substituting $H_i(e^{j\omega})$ in Eq. (10) into Eq. (18), we obtain a set of linear equations as follows;

$$\sum_{n=0}^{N}(-1)^n n^r h_i(n) = 0 \quad (r = 0, 1, \cdots, K-1), \tag{19}$$

where there are K equations with respect to $h_i(n)$.

Moreover, we rewrite the orthonormal condition in Eq. (8) as

$$\sum_{k=0}^{N-2n} h_i(2n+k)h_i(k) = \delta(n) = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} \tag{20}$$

It is clear that there exist $(N+1)/2$ equations in Eq. (20). If $K + L = (N+1)/2$, the number of equations becomes $K + L + (N+1)/2 = N+1$ in Eqs. (16), (19) and (20) with respect to $(N+1)$ unknown coefficients $h_i(n)$. By solving Eqs. (16), (19) and (20), $h_i(n)$ can be obtained for $i = 1, 2$, as proposed in [19]. It should be noted that the orthogonality condition in Eq. (20) is a set of quadratic constraints on the coefficients $h_i(n)$, which is generally difficult to solve. In this paper, we linearize Eq. (20) and use an iterative procedure to solve it (see Appendix).

4. Design of Almost Symmetric Orthogonal Wavelets with Improved Analyticity

In this section, we consider the case of $L + K < (N+1)/2$. The remaining degree of freedom is $I = (N+1)/2 - K - L$. We use the remaining degree of freedom to improve the analyticity.

Let $\hat{\omega}_k(0 < \hat{\omega}_0 < \hat{\omega}_1 < \cdots < \hat{\omega}_{I-1} < \pi)$ be the frequency points at which we want to make the error to be zero;

$$E(\hat{\omega}_k) = H_2(e^{j\hat{\omega}_k}) - H_1(e^{j\hat{\omega}_k})e^{-j(2M+\frac{1}{2})\hat{\omega}_k} = 0, \tag{21}$$

which is separated into real and imaginary parts to get a set of linear equations as follows;

$$\begin{cases} \displaystyle\sum_{n=0}^{N} h_2(n)\cos(n\hat{\omega}_k) - h_1(n)\cos\left(n+2M+\frac{1}{2}\right)\hat{\omega}_k = 0 \\ \displaystyle\sum_{n=0}^{N} h_2(n)\sin(n\hat{\omega}_k) - h_1(n)\sin\left(n+2M+\frac{1}{2}\right)\hat{\omega}_k = 0 \end{cases} \tag{22}$$

for $k = 0, 1, \cdots, I-1$. There are totally $2K + 2L + N + 1 + 2I = 2(N+1)$ equations in Eqs. (16), (19), (20) and (22) with respect to $2(N+1)$ unknown coefficients $h_1(n), h_2(n)$. Therefore, we can obtain $h_1(n)$ and $h_2(n)$ simultaneously by solving Eqs. (16), (19), (20) and (22).

Next, we propose a new design procedure to improve the analyticity of complex wavelet, where the Remez exchange algorithm in [2] is used to minimize the magnitude of the error function $E(\omega)$. The error function $E(\omega)$ has $I+1$ peak points from the coefficients obtained by solving Eqs. (16), (19), (20) and (22). Therefore, we want to make it to be equiripple. In this paper, we apply the Remez exchange algorithm to obtain the equiripple behavior of $E(\omega)$.

Let ω_i $(0 < \omega_0 < \omega_1 < \cdots < \omega_I < \pi)$ be the frequencies of the peak points of $E(\omega)$, which are computed by using the obtained filter coefficients. Then we formulate the error function $E(\omega)$ as follows;

$$E(\omega_i) = H_2(e^{j\omega_i}) - H_1(e^{j\omega_i})e^{-j(2M+\frac{1}{2})\omega_i} = \delta e^{j(\theta_e(\omega_i)+\Delta\theta)} \tag{23}$$

where δ is a magnitude error and $\Delta\theta$ is a phase error. The phase $\theta_e(\omega_i)$ is computed by using the obtained coefficients. Since $\delta e^{j\Delta\theta} = \delta\cos(\Delta\theta) + j\delta\sin(\Delta\theta) = \delta_c + j\delta_s$, Eq. (23) becomes

$$\begin{cases} \sum_{n=0}^{N}\{h_2(n)\cos(n\omega_i) - h_1(n)\cos\left(n+2M+\frac{1}{2}\right)\omega_i\} \\ \qquad -\delta_c\cos(\theta_e(\omega_i)) + \delta_s\sin(\theta_e(\omega_i)) = 0 \\ \sum_{n=0}^{N}\{h_2(n)\sin(n\omega_i) - h_1(n)\sin\left(n+2M+\frac{1}{2}\right)\omega_i\} \\ \qquad -\delta_c\sin(\theta_e(\omega_i)) - \delta_s\cos(\theta_e(\omega_i)) = 0 \end{cases} \tag{24}$$

for $i = 0, 1, \cdots, I$. It should be noted that Eqs. (16), (19), (20) from both of scaling lowpass filters and Eq. (24) have $2K+2L+N+1+2(I+1) = 2N+4$ equations with respect to $2N+2$ coefficients $h_i(n)$ plus δ_c and δ_s. Therefore, we can solve this set of equations to obtain a set of filter coefficients $h_1(n)$ and $h_2(n)$ simultaneously. Furthermore, we make use of an iterative procedure to obtain the equiripple magnitude of $E(\omega)$. Thus, the optimal filter coefficients can be easily obtained through a few iterations. The design algorithm is given in detail as follows.

Design Algorithm {Design of Hilbert Pairs of Almost Symmetric Orthogonal Wavelets}

Begin

1. Read N, K, L and τ_1.
2. Select initial frequency points $\hat{\omega}_k$ $(0 < \hat{\omega}_0 < \hat{\omega}_1 < \cdots < \hat{\omega}_{I-1} < \pi)$ equally spaced in $(0, \pi)$.
3. Solve Eqs. (16), (19), (20) and (22) to obtain a set of initial coefficients $h_1(n), h_2(n)$.
4. Compute $E(\omega)$ to find the peak frequency points Ω_i $(0 < \Omega_0 < \Omega_1 < \cdots < \Omega_I < \pi)$ of $|E(\omega)|$.

Repeat

5) Set $\omega_i = \Omega_i$ $(i = 0, 1, \cdots, I)$.
6) Solve Eqs. (16), (19), (20) and (24) to obtain a set of filter coefficients $h_1(n), h_2(n)$.
7) Compute $E(\omega)$ to find the peak frequency points Ω_i $(0 < \Omega_0 < \Omega_1 < \cdots < \Omega_I < \pi)$ of $|E(\omega)|$.

Until

Satisfy the following condition for a prescribed small constant ϵ (e.g., $\epsilon = 10^{-10}$);

$$\sum_{i=0}^{I}|\omega_i - \Omega_i| < \epsilon$$

End.

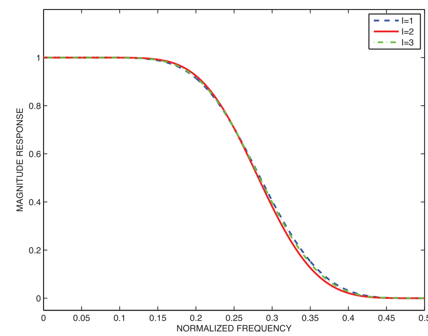


Fig. 1 Magnitude responses of $H_1(z)$ in Example 1.

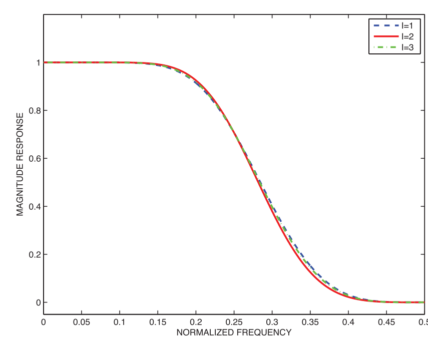


Fig. 2 Magnitude responses of $H_2(z)$ in Example 1.

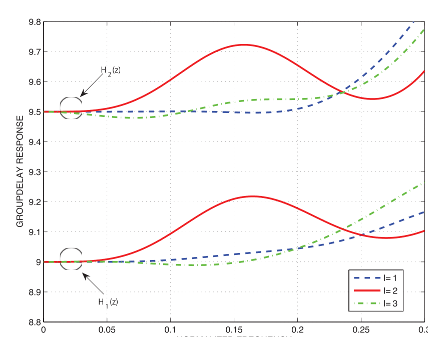


Fig. 3 Group delay responses of $H_1(z)$ and $H_2(z)$ in Example 1.

Example 1: We have used the proposed method to design $H_1(z)$ and $H_2(z)$ with $N = 15$, $K = 4$, and $\tau_1 = 9.0$, $\tau_2 = 9.5$. We choose $\{L, I\} = \{3, 1\}, \{2, 2\}, \{1, 3\}$, respectively. The magnitude responses of three scaling lowpass filters are given in Figs. 1 and 2, respectively. Their group delay responses are shown in Fig. 3, where the half-sample delay condition is approximately achieved. Moreover, the magnitudes of $E(\omega)$ are shown in Fig. 4. It is clear that the euqiripple magnitudes of $E(\omega)$ have been obtained, and the maximum value of $|E(\omega)|$ has been effectively minimized by applying the Remez exchange algorithm. Furthermore, the complex wavelet spectrum $\Psi_c(\omega)$ are given in Fig. 5, which are close to one-sided spectrum. Finally, the analyticity measures of E_∞ and E_2 are summarized in Table 1, where the analyticity of complex wavelets has been improved.

Example 2: We have designed $H_1(z)$ and $H_2(z)$ with

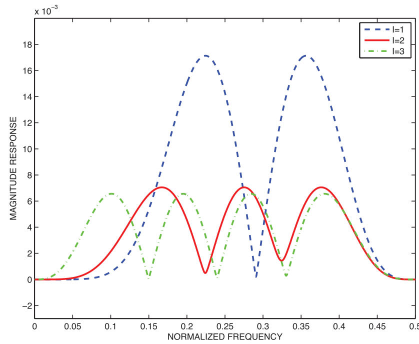


Fig. 4 Magnitude responses of $E(\omega)$ in Example 1.

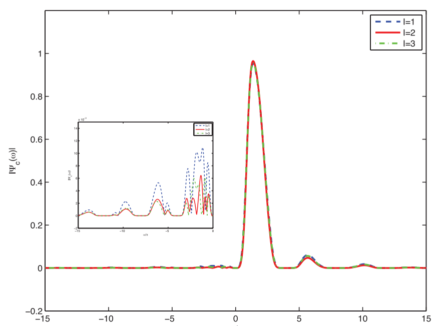


Fig. 5 Magnitude responses of $\Psi_c(\omega)$ in Example 1.

Table 1 Analyticity measures E_∞ and E_2 in Example 1.

L	I	$E_\infty(\%)$	$E_2(\%)$
3	1	1.146	1.360
2	2	0.671	0.569
1	3	0.638	0.665

$N = 21$, $K = 6$, $L = 3$ and $I = 2$. The group delay τ_1 is selected as $\tau_1 = 9.3$, then $\tau_2 = 9.8$ from the half-sample delay condition. The magnitude responses of the scaling lowpass filters $H_i(z)$ are shown in Figs. 6 and 7. For comparison, the magnitude responses of other two filters with $\tau_1 = 8.1, \tau_2 = 8.6$ and $\tau_1 = 11.0, \tau_2 = 11.5$ are shown also in Figs. 6 and 7. The corresponding group delay responses are shown in Fig. 8. Moreover, the magnitudes of $E(\omega)$ are shown in Fig. 9, and are equiripple. It is seen that the maximum value of $|E(\omega)|$ depends on the group delay τ_i too. In addition, the scaling functions $\phi_i(t)$ and wavelet functions $\psi_i(t)$ are given in Fig. 10, respectively. In Fig. 10, the scaling functions have different center of symmetry, while the center of symmetry of wavelet functions remain unchanged. However, the scaling and wavelet functions have different behaviors depending on the group delays. Furthermore, the complex wavelet spectrum $\Psi_c(\omega)$ are given in Fig. 11, and the analyticity measures of E_∞ and E_2 are summarized in Table 2. It is clear that a proper group delay can improve the analyticity.

Example 3: We have designed $H_1(z)$ and $H_2(z)$ with $N = 15, K = 2, L = 3$ and $I = 3$. We set $\tau_1 = 7.25$ and $\tau_2 = 7.75$. For comparison, the Q-shift filter in [12] was also

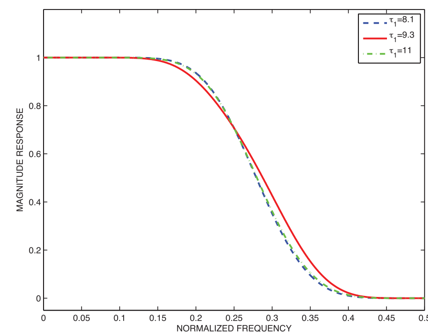


Fig. 6 Magnitude responses of $H_1(z)$ in Example 2.

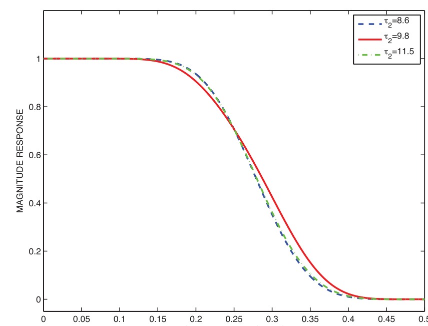


Fig. 7 Magnitude responses of $H_2(z)$ in Example 2.

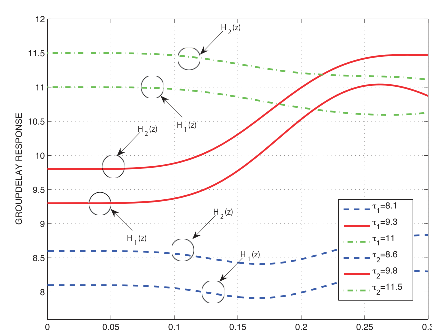


Fig. 8 Group delay responses of $H_1(z)$ and $H_2(z)$ in Example 2.

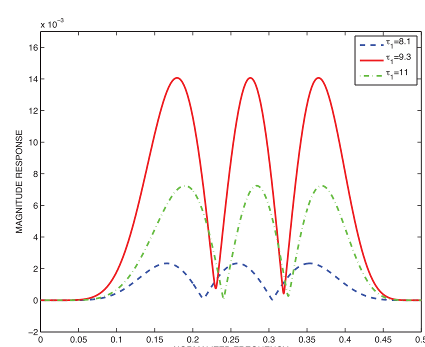
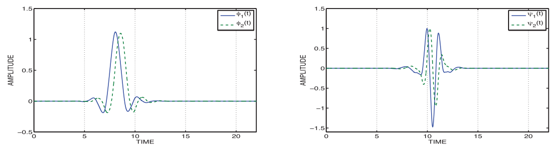


Fig. 9 Magnitude responses of $E(\omega)$ in Example 2.

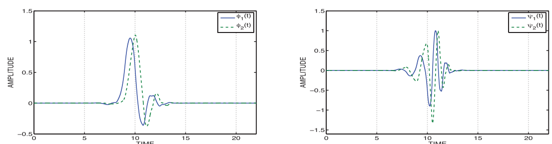
designed, where $N = 15, K = 1, \tau_1 = 7.25, \tau_2 = 7.75$, and AOS/AES filter in [24] where $N = 15, K = 4, \tau_1 = 7.00, \tau_2 = 7.50$. The magnitude responses of the scaling lowpass filters

Table 2 Analyticity measures E_∞ and E_2 in Example 2.

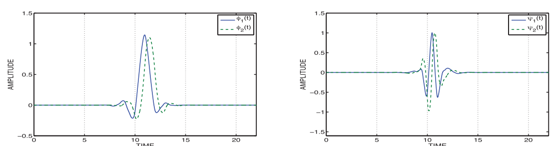
τ_1	τ_2	$E_\infty(\%)$	$E_2(\%)$
8.1	8.6	0.380	0.372
9.3	9.8	1.251	1.092
11.0	11.5	0.740	0.634



(a) $\tau_1 = 8.1, \tau_2 = 8.6$



(b) $\tau_1 = 9.3, \tau_2 = 9.8$



(c) $\tau_1 = 11.0, \tau_2 = 11.5$

Fig. 10 Scaling functions $\phi_i(t)$ and wavelet functions $\psi_i(t)$ in Example 2.

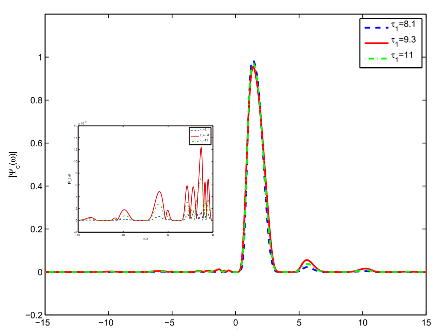


Fig. 11 Magnitude responses of $\Psi_c(\omega)$ in Example 2.

$H_i(z)$ are shown in Figs. 12 and 13, respectively. It is seen that the Q-shift filter has the sharpest magnitude response, but has only one zero at $z = -1$, and AOS/AES filter has four zeros at $z = -1$, then is more flat. In Figs. 12 and 13, the magnitude response of the filter with $\{\tau_1 = 6.50, \tau_2 = 7.00\}$ is also shown. Their group delay responses are shown in Fig. 14, which are consistent with the specified group delays at $\omega = 0$ and more flat than the Q-shift and AOS/AES filters. Moreover, the magnitudes of $E(\omega)$ are shown in Fig. 15, and are smaller than the Q-shift and AOS/AES filters. The complex wavelet spectrum $\Psi_c(\omega)$ are shown in Fig. 16 and the analyticity measures of E_∞ and E_2 are summarized in Table 3. It is clear that the analyticity of the proposed filters are better, compared with the Q-shift and AOS/AES filters.

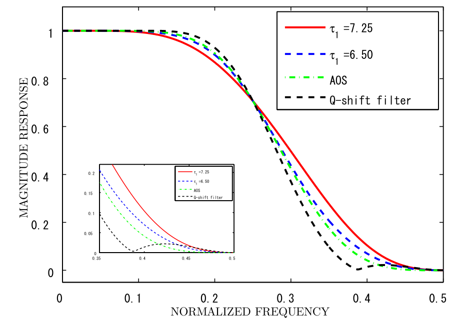


Fig. 12 Magnitude responses of $H_1(z)$ in Example 3.

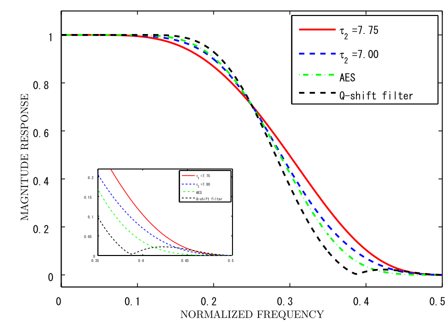


Fig. 13 Magnitude responses of $H_2(z)$ in Example 3.

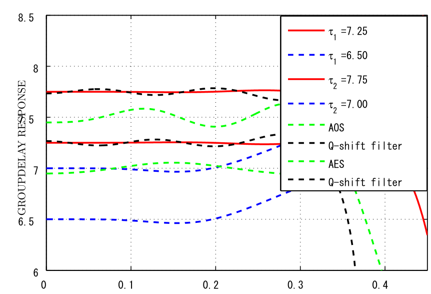


Fig. 14 Group delay responses of $H_1(z)$ and $H_2(z)$ in Example 3.

5. Application to Signal Denoising

One of successful applications of wavelets is signal denoising [7]. In this section, we use the proposed filter banks in application of signal denoising to prove its effectiveness. The test signals used in this experiment are *Blocks*, *Bumps*, *Doppler* and *Heavy Sine* in [7]. The original signal is corrupted by some additive zero-mean white Gaussian noise with variance σ^2. We have generated the noisy signals with $\sigma = 0.4$.

Wavelet thresholding denoising consists of three steps: 1) transform the noisy signal into the wavelet domain to obtain wavelet coefficients, 2) suppress the wavelet coefficients smaller than the given threshold λ, 3) take the inverse transform to reconstruct the denoised signal. In this paper, we have used the hard-thresholding operator, that is, the wavelet coefficient is discarded if its absolute value is smaller than λ. It is shown in [14] that the DTCWT is more efficient

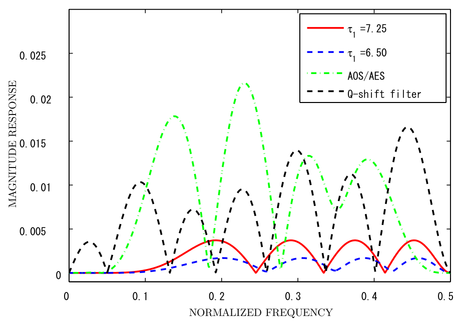


Fig. 15 Magnitude responses of $E(\omega)$ in Example 3.

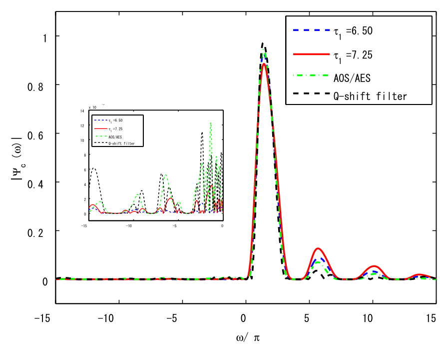


Fig. 16 Magnitude responses of $\Psi_c(\omega)$ in Example 3.

than DWT for denoising, since the DTCWT is nearly shift-invariant, while DWT is lack of shift invariance. We have used the decomposition scheme proposed in [8] and [9]. For the first level of decomposition, the same filter bank of length 8 proposed in [10] was adopted with one sample delay difference. For the rest of levels, we used the proposed filter banks with $\tau_1 = 7.25$ and $\tau_1 = 6.50$ in Example 3, the Q-shift filter with $\tau_1 = 7.25$ and AOS/AES filter with $\tau_1 = 7.00$ for comparison. We have chosen the best threshold λ to obtain the highest SNR. The SNR are shown in Table 4, where the best results are highlighted. It is clear that the filter bank proposed in this paper can achieve a better performance (higher SNR) than the Q-shift and AOS/AES filters. It is because the proposed filter has the improved analyticity.

6. Conclusion

In this paper, we have proposed a new class of Hilbert pairs of almost symmetric orthogonal wavelets. We have specified the degree of flatness of group delay at $\omega = 0$ and the number of vanishing moments, then applied the Remez exchange algorithm to minimize the magnitude of the error function, resulting in the improved analyticity of complex wavelets. Moreover, two scaling lowpass filters can be obtained simultaneously by iteratively solving a set of equations. Therefore, the optimal solution is easily attained through a few iterations. Since the group delay of scaling lowpass filters can be specified arbitrarily, the resulting scaling functions are almost symmetric with an arbitrary center of symmetry.

Table 3 Analyticity measures E_∞ and E_2 in Example 3.

L	I	K	τ_1	τ_2	$E_\infty(\%)$	$E_2(\%)$
Q-shift filter		1	7.25	7.75	1.139	1.338
AOS/AES		4	7.00	7.50	1.310	1.093
3	3	2	6.50	7.00	0.150	0.175
3	3	2	7.25	7.75	0.429	0.441

Table 4 Comparison of SNR (dB) for signal denoising.

Signal	Q-shift filter	AOS/AES	$\tau_1 = 7.25$	$\tau_1 = 6.50$
Blocks	19.79	19.86	19.95	**20.00**
Bumps	23.14	23.10	**23.30**	23.11
Doppler	22.48	22.31	22.06	**22.49**
Heavy Sine	29.54	29.66	30.00	**31.12**
Average	23.74	23.73	23.83	**24.18**

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Appendix: Linearization of Orthogonality Condition

In the design method proposed in this paper, we need to solve a set of nonlinear equations, where only the orthogonality condition in Eq. (20) is a set of quadratic constraints on the filter coefficients $h_i(n)$. In general, it is difficult to solve this nonlinear problem, particularly if the filter is of higher degree, although some nonlinear optimization tools are available, such as Matlab optimization toolbox. In this paper, we linearize the nonlinear equation in Eq. (20), and then use an iterative procedure to obtain a set of filter coefficients, as proposed in [12].

Let $h_i^{(m)}(n)$ be the filter coefficients at mth iteration, which is given by

$$h_i^{(m)}(n) = h_i^{(m-1)}(n) + \Delta h_i^{(m)}(n). \tag{A·1}$$

Then Eq. (20) becomes

$$\sum_{k=0}^{N-2n} h_i^{(m-1)}(k+2n)h_i^{(m-1)}(k) + h_i^{(m-1)}(k+2n)\Delta h_i^{(m)}(k) + h_i^{(m-1)}(k)\Delta h_i^{(m)}(k+2n) + \Delta h_i^{(m)}(k)\Delta h_i^{(m)}(k+2n) = \delta(n). \tag{A·2}$$

Assuming $\Delta h_i^{(m)}(n)$ becomes small enough as m increases, the term $\Delta h_i^{(m)}(k)\Delta h_i^{(m)}(k+2n)$ can be neglected. Eq. (A·2) becomes

$$\sum_{k=0}^{N}\{h_i^{(m-1)}(k+2n) + h_i^{(m-1)}(k-2n)\}\Delta h_i^{(m)}(k) = \delta(n) - \sum_{k=0}^{N-2n} h_i^{(m-1)}(k+2n)h_i^{(m-1)}(k), \tag{A·3}$$

where $h_i^{(m-1)}(n) = 0$ for $n < 0$ and $n > N$. It is clear that Eq. (A·3) is a set of linear equations with respect to $\Delta h_i^{(m)}(n)$, if $h_i^{(m-1)}(n)$ is known. Therefore, we can obtain $\Delta h_i^{(m)}(n)$ by solving the set of linear equations in Eqs. (16), (19), (A·3), and (22) or (24). The filter coefficients $h_i^{(m)}(n)$ are subsequently updated by $\Delta h_i^{(m)}(n)$ as in Eq. (A·1). Therefore, if a proper set of initial coefficients $h_i^{(0)}(n)$ is given, it will converge in the optimal solution (see [19]).

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