Hilbert Transform Pairs of Orthonormal Symmetric Wavelet Bases Using Allpass Filters

Xi ZHANG

Department of Information and Communication Engineering
The University of Electro-Communications

1 Introduction

Hilbert transform pairs of wavelet bases have been proposed and proven to be successful in many signal and image processing applications [1], [2]. In this paper, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases by using allpass filters.

2 Hilbert Transform Pairs of Wavelet Bases

It is well-known that orthonormal wavelet bases can be generated by two-band orthonormal filter banks $\{H_i(z),\ G_i(z)\}$ $(i = 1, 2)$, where $H_i(z)$ are assumed to be lowpass filter, and $G_i(z)$ are highpass. The dilation and wavelet equations give the scaling and wavelet functions;

$$\begin{cases} \phi_i(t) = \sqrt{2}\displaystyle\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\displaystyle\sum_n g_i(n)\phi_i(2t-n) \end{cases}, \tag{1}$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively. It is known in [2] that two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \tag{2}$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ \ \ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \tag{3}$$

if and only if two lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}}, \tag{4}$$

where $\Psi_i(\omega)$ are the Fourier transform of $\psi_i(t)$.

3 Orthonormal Symmetric Solution

In this section, we propose a new class of Hilbert transform pairs of orthonormal symmetric wavelet bases. Firstly, we use a complex allpass filter $A_c(z)$ of order $2N_1$ to construct $H_1(z)$ and $G_1(z)$ [3];

$$\begin{cases} H_1(z) = \dfrac{1}{\sqrt{2}}[A_c(z) + \tilde{A}_c(z)] \\ G_1(z) = \dfrac{z^{-1}}{j\sqrt{2}}[A_c(z) - \tilde{A}_c(z)] \end{cases}, \tag{5}$$

where

$$A_c(z) = z^{-2N_1}e^{j\eta}\frac{\displaystyle\sum_{n=0}^{N_1} a_{2n}^c z^{2n} + j\sum_{n=0}^{N_1-1} a_{2n+1}^c z^{2n+1}}{\displaystyle\sum_{n=0}^{N_1} a_{2n}^c z^{-2n} - j\sum_{n=0}^{N_1-1} a_{2n+1}^c z^{-2n-1}}, \tag{6}$$

where $a_n^c = a_{2N_1-n}^c$ are real, $\eta = \pm\pi/4$ for even N_1 and $\eta = \pm 3\pi/4$ for odd N_1. $\tilde{A}_c(z)$ has a set of coefficients that are complex conjugate with ones of $A_c(z)$. It is seen that $H_1(z)$ and $G_1(z)$ have linear phase responses and satisfy the orthonormality condition. The maximally flat solution for $H_1(z), G_1(z)$ has been proposed in [3], and $H_1(z)$ has $2N_1$ zeros at $z = -1$. Next, we use a real allpass filter $A_r(z)$ of order N_2 to construct $H_2(z)$ and $G_2(z)$ [4];

$$\begin{cases} H_2(z) = \dfrac{1}{\sqrt{2}}[z^K A_r(z^2) + z^{-K-1}A_r(z^{-2})] \\ G_2(z) = \dfrac{1}{\sqrt{2}}[z^K A_r(z^2) - z^{-K-1}A_r(z^{-2})] \end{cases}, \tag{7}$$

where K is integer and $A_r(z)$ is defined by

$$A_r(z) = z^{-N_2}\frac{\displaystyle\sum_{n=0}^{N_2} a_n^r z^n}{\displaystyle\sum_{n=0}^{N_2} a_n^r z^{-n}}, \tag{8}$$

where a_n^r are real. It is known in [4] that $H_2(z)$ and $G_2(z)$ are orthonormal and have linear phase responses also. K must be chosen as $K = 2(N_2 - 2k)$ or $K = 2(N_2 - 2k) - 1$ for $k = 0, 1, \cdots, N_2$. The maximally flat solution for $H_2(z), G_2(z)$ has been given in [4], and $H_2(z)$ has $2N_2 + 1$ zeros at $z = -1$. Therefore, it is clear that $H_1(z)$ and $H_2(z)$ have already satisfied the phase condition in Eq.(4). To satisfy the magnitude condition in Eq.(4) approximately, we choose $N_1 = N_2$ or $N_1 = N_2 + 1$ to ensure $H_1(z)$ and $H_2(z)$ to have a close number of zeros at $z = -1$ as possible. Therefore, the resulting pairs of orthonormal symmetric wavelet bases have almost same degrees of regularity.

4 Conclusion

In this paper, we have proposed a pair of orthonormal symmetric wavelet bases that form the Hilbert transform. The Hilbert transform pairs proposed in this paper have been constructed by using complex and real allpass filters.

References

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