

DESIGN OF HILBERT TRANSFORM PAIRS OF ORTHONORMAL WAVELET BASES WITH IMPROVED ANALYTICITY

Xi ZHANG

Department of Communication Engineering and Informatics
The University of Electro-Communications
1-5-1 Chofugaoka, Chofu-shi, Tokyo, 182-8585 JAPAN
E-mail: xiz@ice.uec.ac.jp

ABSTRACT

This paper proposes a class of Hilbert transform pairs of orthonormal wavelet bases with improved analyticity. To improve the analyticity of complex wavelet, a different allpass filter is used for the half-sample delay approximation. We present a design method for allpass filters with the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band. Remez exchange algorithm is applied in the approximation band, and then a set of filter coefficients can be obtained easily by solving the eigenvalue problem. Therefore, the equiripple phase response is attained through a few iterations. Furthermore, the corresponding filter banks are constructed from the designed allpass filters by using the method proposed in [7]. The resulting orthonormal wavelet bases possess the maximum number of vanishing moments. Finally, one example is presented to demonstrate the improvement of the analyticity.

Keywords: Orthonormal wavelet basis, Hilbert transform pair, FIR filter, Allpass filter, Vanishing moment.

1. INTRODUCTION

Hilbert transform pairs of wavelet bases have been proposed and found to be successful in many applications of signal processing and image processing [3]~[7], [10]. It has been proven in [6], [8] and [9] that the half-sample delay condition between two scaling lowpass filters is the necessary and sufficient condition for the corresponding wavelet bases to form a Hilbert transform pair. Several design procedures for Hilbert transform pairs of wavelet bases have been presented in [3]~[7], [11]. In [7], Selesnick had proposed a class of Hilbert transform pairs of orthonormal wavelet bases, where the corresponding scaling lowpass filters are constructed by using an allpass filter to meet the half-sample delay condition. If the allpass filter is determined, the design is for the scaling lowpass filters to satisfy the condition of orthonormality and to obtain the maximum number of vanishing moments. Thus, this design method is simple and effective. In [7], Selesnick had used the maximally flat allpass filters. However, the maximally flat allpass filters have a larger phase error as ω increases. It will influence the analyticity of complex wavelet, because the accuracy of the half-sample delay approximation is controlled only by the allpass filter.

In this paper, we propose a class of Hilbert transform pairs of orthonormal wavelet bases with improved analyticity. We present a design method of allpass filters for the half-sample delay approximation, which have the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band. We apply Remez exchange algorithm in the approximation band to minimize the

phase error. In the proposed method, a set of filter coefficients can be obtained easily by solving the eigenvalue problem, and then the equiripple phase response is attained through a few iterations. The designed allpass filters are used to improve the analyticity of complex wavelet. The corresponding scaling lowpass filters are constructed by using the method proposed in [7], thus, the resulting orthonormal wavelet bases possess the maximum number of vanishing moments. Finally, one example is presented to demonstrate the improvement of the analyticity.

2. HILBERT TRANSFORM PAIRS OF WAVELET BASES

It is well-known that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z), G_i(z)\}$, where $i = 1, 2$. Now we assume that $H_i(z)$ and $G_i(z)$ are lowpass and highpass filters, respectively. The condition of orthonormality for $H_i(z)$ and $G_i(z)$ is given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0 \end{cases} \quad (1)$$

Let $\phi_i(t)$, $\psi_i(t)$ be the corresponding scaling and wavelet functions, respectively. The dilation and wavelet equations give the scaling and wavelet functions;

$$\begin{cases} \phi_i(t) = \sqrt{2} \sum_n h_i(n) \phi_i(2t - n) \\ \psi_i(t) = \sqrt{2} \sum_n g_i(n) \phi_i(2t - n) \end{cases}, \quad (2)$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively.

It has been proven in [6], [8] and [9] that two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \quad (3)$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \quad (4)$$

if and only if two scaling lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \quad (-\pi < \omega < \pi), \quad (5)$$

where $\Psi_i(\omega)$ are the Fourier transform of $\psi_i(t)$. This is the so-called half-sample delay condition between two scaling lowpass filters. Eq.(5) is the necessary and sufficient condition for two orthonormal wavelet bases to form a Hilbert transform pair.

3. HILBERT TRANSFORM PAIRS OF ORTHONORMAL WAVELET BASES COMPOSED OF ALLPASS FILTERS

The transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-L} \frac{D(z^{-1})}{D(z)}, \quad (6)$$

where

$$D(z) = \sum_{n=0}^L d(n)z^{-n}, \quad (7)$$

L is the degree of $A(z)$, $d(n)$ are real coefficients and $d(0) = 1$.

In [7], Selesnick had proposed that the scaling lowpass filters $H_1(z)$ and $H_2(z)$ have the following form;

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}) \end{cases}, \quad (8)$$

and $G_i(z) = z^{-M}H_i(-z^{-1})$ for $i = 1, 2$, where M is the degree of $H_i(z)$ and is an odd number.

Since both of scaling lowpass filters have the same component $F(z)$, we have

$$H_2(z) = H_1(z)z^{-L} \frac{D(z^{-1})}{D(z)} = H_1(z)A(z). \quad (9)$$

Therefore, if $A(z)$ in Eq.(6) is an approximate half-sample delay;

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}} \quad (-\pi < \omega < \pi), \quad (10)$$

then the half-sample delay condition in Eq.(5) is achieved approximately. Thus, two wavelet bases form an approximate Hilbert transform pair.

Once $A(z)$ is determined, $F(z)$ needs to be designed for $H_1(z)$ and $H_2(z)$ to satisfy the condition of orthonormality and to have the maximum number of vanishing moments.

To obtain wavelet bases with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1 + z^{-1})^K. \quad (11)$$

Thus

$$\begin{cases} H_1(z) = Q(z)(1 + z^{-1})^K D(z) \\ H_2(z) = Q(z)(1 + z^{-1})^K z^{-L} D(z^{-1}) \end{cases}. \quad (12)$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(1 + z)^K(1 + z^{-1})^K D(z)D(z^{-1}). \end{aligned} \quad (13)$$

Defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-N}^N r(n)z^{-n}, \quad (14)$$

$$S(z) = (z + 2 + z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \quad (15)$$

where $r(n) = r(-n)$ for $1 \leq n \leq N$ and $s(n) = s(-n)$ for $1 \leq n \leq L + K$, then we have

$$P(z) = R(z)S(z). \quad (16)$$

Note that $P(z)$ is a halfband filter, thus, the degree of $H_i(z)$ is $M = N + L + K$ and M is an odd number.

We can write the condition of orthonormality in Eq.(1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n - k)r(k) = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases}, \quad (17)$$

where $I_{min} = \max\{-N, 2n - L - K\}$ and $I_{max} = \min\{N, 2n + L + K\}$. In Eq.(17), there are $(M + 1)/2$ equations with respect to $(N + 1)$ unknown coefficients $r(n)$. Therefore, it is clear that we can obtain the only solution if $(M + 1)/2 = N + 1$. Given N and L , the maximal K is $K_{max} = N - L + 1$, then the obtained filters have the maximally flat magnitude responses. That is, the corresponding wavelet bases have the maximum number of vanishing moments. In other words, the minimal degree of $Q(z)$ is $N_{min} = L + K - 1$ for given L and K , as shown in [7]. In addition, the scaling lowpass filters with improved magnitude responses have been also proposed by using Remez exchange algorithm in [11].

4. HILBERT TRANSFORM PAIRS OF WAVELET BASES WITH IMPROVED ANALYTICITY

It is known that if two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ are a pair of Hilbert transform, the complex wavelet $\psi_1(t) + j\psi_2(t)$ is analytic, i.e., its spectrum is one-sided: $\Psi_1(\omega) + j\Psi_2(\omega) = 0$ for $\omega < 0$. However, it cannot be exact in practice, because the half-sample delay condition in Eq.(5) can only be approximated with real filters. In this section, we will discuss how to improve its analyticity.

It is seen in Eq.(5) that $H_2(e^{j\omega})$ needs to be approximated to $H_1(e^{j\omega})e^{-j\frac{\omega}{2}}$. We define the error function $E(\omega)$ as

$$E(\omega) = H_2(e^{j\omega}) - H_1(e^{j\omega})e^{-j\frac{\omega}{2}}. \quad (18)$$

From Eq.(9), we have

$$E(\omega) = H_1(e^{j\omega})[A(e^{j\omega}) - e^{-j\frac{\omega}{2}}], \quad (19)$$

thus

$$|E(\omega)| = 2|H_1(e^{j\omega})| \left| \sin \frac{\theta(\omega) + \frac{\omega}{2}}{2} \right|, \quad (20)$$

where $\theta(\omega)$ is the phase response of $A(z)$. It is clear from Eq.(20) that $|E(\omega)|$ is dependent on both the magnitude response $|H_1(e^{j\omega})|$ and the phase error of $A(z)$. Since $H_1(z)$ is a lowpass filter, we must minimize the phase error not only in passband but also in transition band to improve the analyticity of complex wavelet.

In [7], Selesnick had used the maximally flat allpass filters. Since $\omega = 0$ is chosen as the point of approximation, the phase error will increase as ω goes away from $\omega = 0$. Thus, $|E(\omega)|$ has a large error in transition band (see Fig.2). There are many design methods for allpass filters to approximate a fractional delay, e.g., maximally flat, least square [1], equiripple approximation [2], and so on. It is known that the wavelet function is defined by the infinite product formula. Thus, it is necessary that $A(z)$ has a certain degree of flatness at $\omega = 0$ to improve the analyticity. In the following, we present a design method of allpass filters with the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band.

Let $\theta_d(\omega) = -\frac{1}{2}\omega$ be the desired phase response. The difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d(\omega)$ is

$$\theta_e(\omega) = \theta(\omega) - \theta_d(\omega) = 2 \tan^{-1} \frac{N_L(\omega)}{D_L(\omega)}, \quad (21)$$

where

$$\begin{cases} N_L(\omega) = \sum_{n=0}^L d(n) \sin\{(n - \frac{L}{2} + \frac{1}{4})\omega\} \\ D_L(\omega) = \sum_{n=0}^L d(n) \cos\{(n - \frac{L}{2} + \frac{1}{4})\omega\} \end{cases}. \quad (22)$$

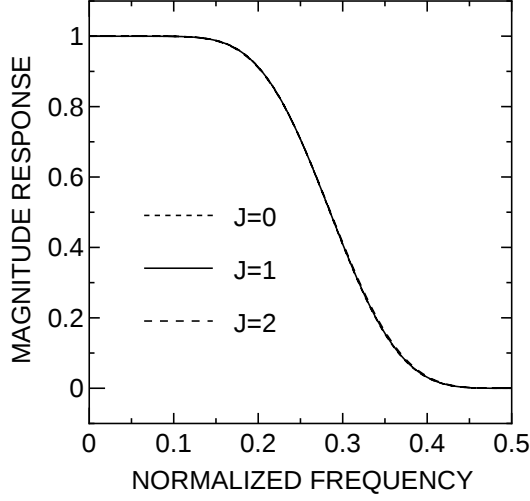


Fig. 1. Magnitude responses of scaling lowpass filters $H_i(z)$.

Therefore, the design problem is to satisfy the flatness condition and to minimize the phase error $\theta_e(\omega)$ in the approximation band.

Firstly, we consider the flatness condition of the phase response at $\omega = 0$. It is required that the derivatives of $\theta(\omega)$ are equal to that of $\theta_d(\omega)$ at $\omega = 0$;

$$\left. \frac{\partial^{2r+1}\theta(\omega)}{\partial\omega^{2r+1}} \right|_{\omega=0} = \left. \frac{\partial^{2r+1}\theta_d(\omega)}{\partial\omega^{2r+1}} \right|_{\omega=0} \quad (r = 0, 1, \dots, J-1), \quad (23)$$

where J is a parameter that controls the degree of flatness, and $0 \leq J \leq L$. Eq.(23) is equivalent to

$$\left. \frac{\partial^{2r+1}\theta_e(\omega)}{\partial\omega^{2r+1}} \right|_{\omega=0} = 0 \quad (r = 0, 1, \dots, J-1). \quad (24)$$

By using Eq.(21), Eq. (24) can be reduced to

$$\left. \frac{\partial^{2r+1}N_L(\omega)}{\partial\omega^{2r+1}} \right|_{\omega=0} = 0 \quad (r = 0, 1, \dots, J-1). \quad (25)$$

By substituting $N_L(\omega)$ in Eq.(22) into Eq.(25), we derive a system of linear equations as follows;

$$\sum_{n=0}^L \left(n - \frac{L}{2} + \frac{1}{4}\right)^{2r+1} d(n) = 0 \quad (r = 0, 1, \dots, J-1). \quad (26)$$

If $J = L$, we can solve the linear equations in Eq.(26) to obtain the maximally flat allpass filters, due to $d(0) = 1$.

Next, we consider the case of $J < L$. We want to obtain an equiripple phase response in the approximation band $[0, \omega_c]$ by using the remaining degree of freedom. Let ω_i ($0 < \omega_0 < \omega_1 < \dots < \omega_{L-J} = \omega_c$) are the extremal frequencies in the approximation band. We apply the Remez exchange algorithm and formulate $\theta_e(\omega)$ as

$$\tan \frac{\theta_e(\omega_i)}{2} = \frac{\sum_{n=0}^L d(n) \sin\{(n - \frac{L}{2} + \frac{1}{4})\omega_i\}}{\sum_{n=0}^L d(n) \cos\{(n - \frac{L}{2} + \frac{1}{4})\omega_i\}} = (-1)^i \delta, \quad (27)$$

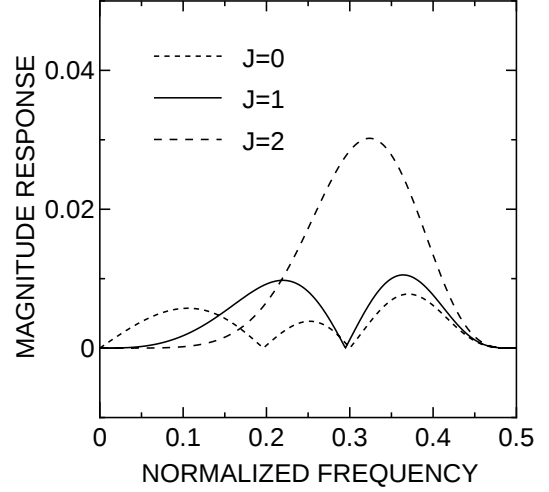


Fig. 2. Magnitude responses of $E(\omega)$.

where δ is an error. Then we rewrite Eqs.(26) and (27) in the matrix form as

$$\mathbf{P}\mathbf{x} = \delta\mathbf{Q}\mathbf{x}, \quad (28)$$

where $\mathbf{x} = [d(0), d(1), \dots, d(L)]^T$, and the elements of the matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$P_{mn} = \begin{cases} (n - \frac{L}{2} + \frac{1}{4})^{(2m+1)} & (m = 0, 1, \dots, J-1) \\ \sin\{(n - \frac{L}{2} + \frac{1}{4})\omega_{(m-J)}\} & (m = J, J+1, \dots, L) \end{cases} \quad (29)$$

$$Q_{mn} = \begin{cases} 0 & (m = 0, 1, \dots, J-1) \\ (-1)^{(m-J)} \cos\{(n - \frac{L}{2} + \frac{1}{4})\omega_{(m-J)}\} & (m = J, J+1, \dots, L) \end{cases} \quad (30)$$

Therefore, it should be noted that Eq.(28) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\mathbf{x}$ is a corresponding eigenvector. To minimize δ , we must find the absolute minimum eigenvalue by solving the eigenvalue problem, so that the corresponding eigenvector gives a set of filter coefficients $d(n)$. We make use of an iteration procedure to obtain the equiripple phase response. The design algorithm is shown as follows.

Procedure {Allpass Filter Design Algorithm}

Begin

- 1) Read L , J , and ω_c .
- 2) Select initial extremal frequencies Ω_i ($0 < \Omega_0 < \Omega_1 < \dots < \Omega_{L-J} = \omega_c$) equally spaced in $[0, \omega_c]$.

Repeat

- 3) Set $\omega_i = \Omega_i$ ($i = 0, 1, \dots, L-J$).
- 4) Compute $\mathbf{P}$ and $\mathbf{Q}$, then find the absolute minimum eigenvalue δ to obtain a set of filter coefficients $d(n)$.
- 5) Search the peak frequencies Ω_i ($0 < \Omega_0 < \Omega_1 < \dots < \Omega_{L-J} = \omega_c$) of $\theta_e(\omega)$ in $[0, \omega_c]$.

Until

Satisfy the following condition for a prescribed small constant ϵ :

$$|\omega_i - \Omega_i| < \epsilon \quad (\text{for all } i)$$

End.

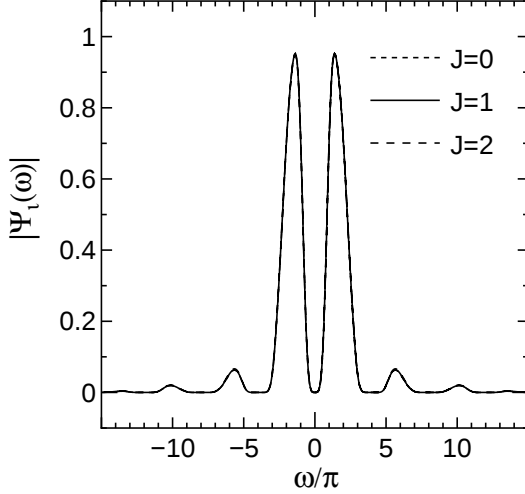


Fig. 3. Magnitude responses of $\Psi_i(\omega)$.

5. DESIGN EXAMPLE

In this section, we present one example to demonstrate the improvement of the analyticity. Firstly, we have designed an allpass filter with $L = 2$, $J = 1$, $\omega_c = \frac{5}{8}\pi$, and used the method proposed in [7] to construct the scaling lowpass filters $H_i(z)$ with $N = 5$, $K = 4$. Its magnitude response is shown in solid line in Fig.1. For comparison, the magnitude responses of $H_i(z)$ using different allpass filters with $J = 0$ and $J = 2$ are shown in Fig.1 also. Note that $J = 2$ means the maximally flat allpass filter, while $J = 0$ is the equiripple allpass filter without the flatness condition. These scaling lowpass filters have the same degree ($M = 11$), and their wavelet bases have the maximum number of vanishing moments. It is seen in Fig.1 that the magnitude responses are almost same. However, $E(\omega)$ are different, as shown in Fig.2. When $J = 0$, the maximum error of $E(\omega)$ is minimum, while it is maximum when $J = 2$. Moreover, the spectrum $\Psi_i(\omega)$ of the resulting wavelet functions $\psi_i(t)$ are shown in Fig.3, and are almost same. The spectrum $\Psi_1(\omega) + j\Psi_2(\omega)$ of the complex wavelet $\psi_1(t) + j\psi_2(t)$ are shown in Fig.4, which approximate zero for $\omega < 0$. We define the peak error as $E_1 = \max_{\omega < 0} |\Psi_1(\omega) + j\Psi_2(\omega)| / \max_{\omega > 0} |\Psi_1(\omega) + j\Psi_2(\omega)|$. When $J = 1$, $E_1 = 0.00771$ is minimum, while when $J = 0, 2$, $E_1 = 0.00877$, and $E_1 = 0.01590$, respectively. It is clear that the analyticity have been improved.

6. CONCLUSION

In this paper, we have proposed a class of Hilbert transform pairs of orthonormal wavelet bases with improved analyticity. To improve the analyticity of complex wevelet, we have presented a design method for allpass filters with the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band. We have applied Remez exchange algorithm in the approximation band to minimize the phase error. Therefore, a set of filter coefficients can be obtained easily by solving the eigenvalue problem, and the equiripple phase response is attained through a few iterations. The resulting orthonormal wavelet bases possess the maximum number of vanishing moments as proposed in the conventional methods. Finally, we have presented one example to demonstrate the improvement of the analyticity.

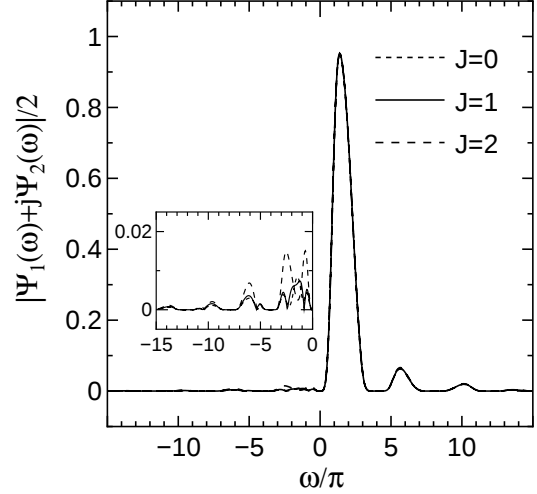


Fig. 4. Magnitude responses of $(\Psi_1(\omega) + j\Psi_2(\omega))/2$.

7. REFERENCES

- [1] T. I. Laakso, V. Valimaki, M. Karjalainen and U. K. Laine, "Splitting the unit delay: Tools for fractional delay filter design," IEEE Signal Processing Mag., vol.13, no.1, pp.30–60, Jan. 1996.
- [2] X. Zhang and H. Iwakura, "Design of IIR digital allpass filters based on eigenvalue problem," IEEE Trans. Signal Processing, vol.47, no.2, pp.554–559, Feb. 1999.
- [3] N. G. Kingsbury, "The dual-tree complex wavelet transform: A new technique for shift invariance and directional filters," in Proc. 8th IEEE DSP Workshop, Utan, no.86, Aug. 1998.
- [4] N. G. Kingsbury, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in Proc. ICIP2000, Vancouver, Canada, vol.2, pp.375–378, Sep. 2000.
- [5] N. G. Kingsbury, "Complex wavelets for shift invariant analysis and filtering of signals," Appl. Comput. Harmon. Anal., vol.10, no.3, pp.234–253, May 2001.
- [6] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," IEEE Signal Processing Letters, vol.8, no.6, pp.170–173, Jun. 2001.
- [7] I. W. Selesnick, "The design of approximate Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.50, no.5, pp.1144–1152, May 2002.
- [8] H. Ozkaramanli and R. Yu, "On the phase condition and its solution for Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.51, no.12, pp.3293–3294, Dec. 2003.
- [9] R. Yu and H. Ozkaramanli, "Hilbert transform pairs of orthogonal wavelet bases: Necessary and sufficient conditions," IEEE Trans. Signal Processing, vol.53, no.12, pp.4723–4725, Dec. 2003.
- [10] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The dual-tree complex wavelet transform," IEEE Signal Processing Mag., vol.22, no.6, pp.123–151, Nov. 2005.
- [11] X. Zhang, "Design of Hilbert transform pairs of orthonormal wavelet bases using Remez exchange algorithm," in Proc. ICIP2009, Cairo, Egypt, pp.3813–3816, Nov. 2009.