# DESIGN OF CAUSAL IIR PERFECT RECONSTRUCTION FILTER BANKS

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## ABSTRACT

This paper presents a new method for designing two channel biorthogonal IIR filter banks, which satisfy both the perfect reconstruction and causal stable conditions. The proposed method is based on the formulation of a generalized eigenvalue problem by using Remez multiple exchange algorithm. Therefore, the filter coefficients can be computed by solving the eigenvalue problem, and the optimal solution is easily obtained through a few iterations. One example is designed to demonstrate the effectiveness of the proposed method.

## 1. INTRODUCTION

Two channel perfect reconstruction (PR) filter banks have been used in different applications of signal processing [1]. The theory and design of FIR PR filter banks have been well established in recent years. In this paper, we will consider design of two channel IIR PR filter banks that satisfy the causal stable condition. A class of two channel causal IIR PR filter banks have been proposed in [2] and [3]. In [3], the proposed filter banks are based on general IIR filters, but the obtained magnitude responses are poor. In [2], an efficient structurally perfect reconstruction implementation is presented, where for IIR case, allpass filters are used. There exists, however, a bump of approximately $4dB$ at $\omega = \pi/2$.

In this paper, we propose a new design method for two channel biorthogonal IIR PR filter banks that satisfy the causal stable condition. We adopt the structurally perfect reconstruction implementation proposed in [3], but use general IIR filters rather than allpass filters of [3]. By using general IIR filters, the bump around $\omega = \pi/2$ caused when allpass filters are uesd can be suppressed. To obtain the optimal solution in the Chebyshev sense, we apply Remez multiple exchange algorithm and formulate the design problem in the form of a generalized eigenvalue problem [4],[5]. Then, the filter coefficients can be computed by solving the eigenvalue problem to get the positive minimum eigenvalue, and the optimal solution is easily obtained through a few iterations. Finally, one design example is presented to demonstrate the effectiveness of the proposed method.

## 2. BIORTHOGONAL IIR FILTER BANKS

In two channel filter banks, assume that $H_0(z), H_1(z)$ are analysis filters, and $G_0(z), G_1(z)$ are synthesis filters. It is well-known that the perfect reconstruction condition is

$$\begin{cases} G_0(z) = H_1(-z) \\ G_1(z) = -H_0(-z) \\ H_0(z)H_1(-z) - H_1(z)H_0(-z) = z^{-2K-1} \end{cases}, \quad (1)$$

where $K$ is integer. In [3], the analysis filters $H_0(z)$ and $H_1(z)$ are composed by

$$\begin{cases} H_0(z) &= \frac{1}{2}\{z^{-2N-1} + A(z^2)\} \\ H_1(z) &= z^{-2M} - B(z^2)H_0(z) \\ &= z^{-2M} - \frac{B(z^2)}{2}\{z^{-2N-1} + A(z^2)\} \end{cases}, \quad (2)$$

where $N$ and $M$ are integers. Then the perfect reconstruction condition of Eq.(1) can be satisfied. The structurally perfect reconstruction implementation proposed in [3] is shown in Fig.1. In [3], for IIR case, $A(z)$ and $B(z)$ employ the same allpass filter, however, there exists a bump

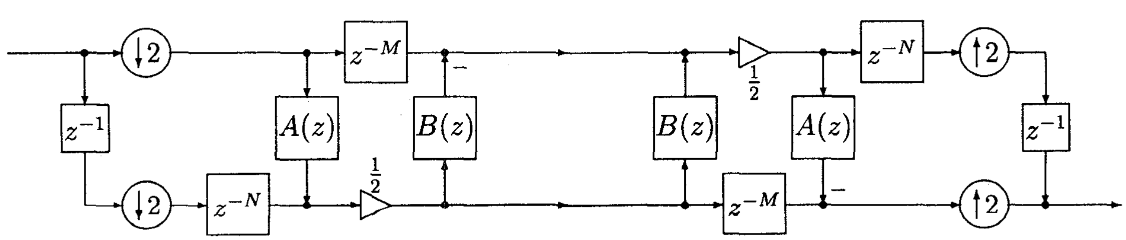


**Fig.1** Structurally perfect reconstruction filter bank.

of approximately $4dB$ at $\omega = \pi/2$. In this paper, we use general IIR filters rather than allpass filters, i.e.,

$$\begin{cases} A(z) = \sum_{i=0}^{L_1} a_i z^{-i} / \sum_{i=0}^{L_2} b_i z^{-i} \\ B(z) = \sum_{i=0}^{L_3} c_i z^{-i} / \sum_{i=0}^{L_4} d_i z^{-i} \end{cases}, \tag{3}$$

where $L_1, L_2, L_3, L_4$ are integers, the filter coefficients $a_i, b_i, c_i, d_i$ are real, and $b_0 = d_0 = 1$.

## 3. DESIGN OF IIR PR FILTER BANKS

In this section, we describe design of IIR PR filter banks based on eigenvalue problem by using Remez multiple exchange algorithm.

### 3.1. Desired Frequency Responses

From Eq.(2), we have

$$H_0(z) = \frac{z^{-2N-1}}{2}\{1+\frac{A(z^2)}{z^{-2N-1}}\} = \frac{z^{-2N-1}}{2}\{1+\hat{A}(z^2)\}, \tag{4}$$

where $\hat{A}(z^2) = z^{2N+1}A(z^2)$. Then, the desired frequency response of $\hat{A}(z^2)$ is

$$\begin{cases} \hat{A}(e^{j2\omega}) = 1 & (0 \le \omega \le \omega_p) \\ \hat{A}(e^{j2\omega}) = -1 & (\omega_s \le \omega \le \pi) \end{cases}, \tag{5}$$

where $\omega_p, \omega_s$ are the passband and stopband edge frequencies respectively, and $\omega_p + \omega_s = \pi$. Since $\hat{A}(e^{j2(\pi-\omega)}) = -\hat{A}^*(e^{j2\omega})$, thus the desired frequency response of $A(z)$ is

$$A_d(e^{j\omega}) = e^{-j(N+\frac{1}{2})\omega} \qquad (0 \le \omega \le 2\omega_p), \tag{6}$$

where $x^*$ denotes the complex conjugate of $x$. It can be seen from Eq.(2) that in $[\omega_s, \pi]$, $H_1(e^{j\omega}) = e^{-j2M\omega}$ since $H_0(e^{j\omega}) = 0$, i.e., the magnitude of $H_1(z)$ is 1 and the phase is linear. In $[0, \omega_p]$, ideally, $H_0(z) = z^{-2N-1}$, then,

$$H_1(z) = z^{-2M} - B(z^2)z^{-2N-1} = z^{-2M}\{1 - \hat{B}(z^2)\}, \tag{7}$$

where $\hat{B}(z^2) = z^{2(M-N)-1}B(z^2)$. To force $H_1(e^{j\omega})$ to be 0 in $[0, \omega_p]$, the desired frequency response of $\hat{B}(z^2)$ is

$$\hat{B}(e^{j2\omega}) = 1 \qquad (0 \le \omega \le \omega_p), \tag{8}$$

thus the desired frequency response of $B(z)$ is

$$B_d(e^{j\omega}) = e^{-j(M-N-\frac{1}{2})\omega} \qquad (0 \le \omega \le 2\omega_p). \tag{9}$$

Therefore, the design problem will become the complex Chebyshev approximation of $A(z)$ and $B(z)$.

### 3.2. Design of $H_0(z)$

Here, we consider design of $H_0(z)$, i.e., $A(z)$. We define an error function between the frequency response and the desired frequency response of $A(z)$ as

$$E_a(\omega) = \frac{A(e^{j\omega}) - A_d(e^{j\omega})}{e^{-j(N+\frac{1}{2})\omega}} = \hat{A}(e^{j\omega}) - 1. \tag{10}$$

#### *3.2.1. Formulation*

To apply Remez multiple exchange algorithm, we first select $J+1$ ($J = \lfloor \frac{L_1+L_2+1}{2} \rfloor$) extremal frequencies $\omega_i$ ($2\omega_p = \omega_0 > \omega_1 > \cdots > \omega_J \ge 0$), where $\omega_J > 0$ when $L_1 + L_2$ is even, and $\omega_J = 0$ when $L_1 + L_2$ is odd. $\lfloor x \rfloor$ denotes the integer part of $x$. Then, we can formulate $E_a(\omega)$ as follows;

$$E_a(\omega_i) = \hat{A}(e^{j\omega_i}) - 1 = \delta e^{j\theta(\omega_i)}, \tag{11}$$

where $\delta$ ($> 0$) is magnitude error to be minimized, and $\theta(\omega_i)$ is phase response at $\omega_i$ and can be computed in the previous iteration. Substituting $E_a(\omega)$ of Eq.(10) into Eq.(11), we divide Eq.(11) into the real and imaginary parts as

$$\begin{aligned} \sum_{m=0}^{L_1} a_m \cos(N-m+\frac{1}{2})\omega_i - \sum_{m=0}^{L_2} b_m \cos(m\omega_i) \\ = \delta \sum_{m=0}^{L_2} b_m \cos(\theta(\omega_i) - m\omega_i) \\ (i = 0, 1, \cdots, J), \end{aligned} \tag{12}$$

$$\begin{aligned} \sum_{m=0}^{L_1} a_m \sin(N-m+\frac{1}{2})\omega_i + \sum_{m=0}^{L_2} b_m \sin(m\omega_i) \\ = \delta \sum_{m=0}^{L_2} b_m \sin(\theta(\omega_i) - m\omega_i) \\ (i = 0, 1, \cdots, J_1), \end{aligned} \tag{13}$$

where $J_1 = J$ when $L_1 + L_2$ is even, and $J_1 = J - 1$ when $L_1 + L_2$ is odd, since Eq.(11) has not imaginary part at $\omega_J = 0$. Eqs.(12) and (13) can be rewritten in the matrix form as

$$PA = \delta QA, \tag{14}$$

where $A = [a_0, a_1, \cdots, a_{L_1}, b_0, b_1, \cdots, b_{L_2}]^T$,

$$P = \begin{bmatrix} \cos(N+\frac{1}{2})\omega_0 & \cdots & -1 & \cdots & -\cos L_2\omega_0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \cos(N+\frac{1}{2})\omega_J & \cdots & -1 & \cdots & -\cos L_2\omega_J \\ \sin(N+\frac{1}{2})\omega_0 & \cdots & 0 & \cdots & \sin L_2\omega_0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \sin(N+\frac{1}{2})\omega_{J_1} & \cdots & 0 & \cdots & \sin L_2\omega_{J_1} \end{bmatrix} \tag{15}$$

$$Q = \begin{bmatrix} 0 & \cdots & \cos\theta(\omega_0) & \cdots & \cos(\theta(\omega_0) - L_2\omega_0) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cos\theta(\omega_J) & \cdots & \cos(\theta(\omega_J) - L_2\omega_J) \\ 0 & \cdots & \sin\theta(\omega_0) & \cdots & \sin(\theta(\omega_0) - L_2\omega_0) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sin\theta(\omega_{J_1}) & \cdots & \sin(\theta(\omega_{J_1}) - L_2\omega_{J_1}) \end{bmatrix} \tag{16}$$

It should be noted that Eq.(14) corresponds to a generalized eigenvalue problem, i.e., $\delta$ is an eigenvalue and $A$ is a corresponding eigenvector. Therefore, to minimize the magnitude error $\delta$, we must compute the positive minimum eigenvalue by solving the above eigenvalue problem [4],[5]. Then, the corresponding eigenvector gives a set of filter coefficients. By appropriately selecting the initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$, we can apply an iteration procedure to attain the optimal solution in the Chebyshev sense. The selection of the initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$ will directly influence convergence of the iteration procedure. In the following, we will discuss how to select the initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$.

#### *3.2.2. Selection of Initial Value*

In the proposed iteration procedure, arbitrarily selecting an initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$ cannot guarantee to converge to the optimal solution. Hence, it is very important how to select the initial value. We assume that there exist $J_1+1$ frequency points $\bar{\omega}_i$ $(2\omega_p > \bar{\omega}_0 > \bar{\omega}_1 > \cdots > \bar{\omega}_{J_1} \geq 0)$ so that the error function $E_a(\omega)$ satisfies

$$E_a(\bar{\omega}_i) = \hat{A}(e^{j\bar{\omega}_i}) - 1 = 0, \tag{17}$$

where $\bar{\omega}_{J_1} = 0$ when $L_1+L_2$ is even, and $\bar{\omega}_{J_1} > 0$ when $L_1+L_2$ is odd. A possible choice of $\bar{\omega}_i$ is to pick these frequency points equally spaced in $[0, 2\omega_p]$. Other distributions may also be prefered to decrease number of iterations. Since $b_0 = 1$, Eq.(17) can be rewritten into

$$\sum_{m=0}^{L_1} a_m \cos(N - m + \frac{1}{2})\bar{\omega}_i - \sum_{m=1}^{L_2} b_m \cos(m\bar{\omega}_i) = 1 \qquad (i = 0, 1, \cdots, J_1), \tag{18}$$

$$\sum_{m=0}^{L_1} a_m \sin(N - m + \frac{1}{2})\bar{\omega}_i + \sum_{m=1}^{L_2} b_m \sin(m\bar{\omega}_i) = 0 \qquad (i = 0, 1, \cdots, J-1), \tag{19}$$

which is a set of linear equations. Hence, we can obtain an initial solution of filter coefficients by solving the linear equations of Eqs.(18) and (19). Then, we compute the error function $E_a(\omega)$ by using the obtained filter coefficients, and search for the peak points of $|E_a(\omega)|$ in the band $[0, 2\omega_p]$ to get $J+1$ initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$. The design algorithm is shown as follows.

#### *3.2.3. Design Algorithm*

**Procedure** {Design Algorithm of IIR PR Filter Banks}

**Begin**

1. Read $L_1, L_2, N$ and $\omega_p$.
2. Select $J_1 + 1$ frequency points $\bar{\omega}_i$ equally spaced in the band $[0, 2\omega_p]$.
3. Solve Eqs.(18) and (19) to obtain an initial solution of filter coefficients $a_i$ and $b_i$.
4. Compute error function $E_a(\omega)$ by using the initial filter coefficients, then search peak frequencies as initial extremal frequencies $\Omega_i$ and compute its phase $\theta(\Omega_i)$.

**Repeat**

5. Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, J$.
6. Compute $P$ and $Q$ by using Eqs.(15) and (16), then find the positive minimum eigenvalue of Eq.(14) to obtain a set of filter coefficients $a_i$ and $b_i$.
7. Compute error function $E_a(\omega)$, then search peak frequencies $\Omega_i$ and compute its phase $\theta(\Omega_i)$.

**Until** Satisfy the following condition for a prescribed small constant $\epsilon$ :

$$\{|\Omega_i - \omega_i| \leq \epsilon \qquad (\text{for } i = 0, 1, \cdots, J)\}$$

8. Check stability of $A(z)$ by finding the locations of poles.

**End**.

#### *3.2.4. Stable Condition*

In the above design algorithm, the obtained filter $A(z)$ may not be guaranteed to be stable. The stability of $A(z)$ must be checked in step.8 by computing the locations of the poles. The stability of $A(z)$ are generally dependent on the specifications, i.e., $L_1$, $L_2$ and $N$. When $L_1$ and $L_2$ are given, the group delay must be chosen enough large to guarantee the obtained filter to be stable, i.e., $N \geq N_{min}$, where $N_{min}$ is the minimum group delay for the stable filters. In our experience, $N_{min}$ is directly proportional to $L_1$ and $L_2$, in general.

### 3.3. Design of $H_1(z)$

Here, we consider design of $H_1(z)$, i.e., $B(z)$. We can design $B(z)$ by using the design method of $A(z)$ proposed in **3.2**. However, it can be seen from Eq.(2) that the frequency response of $H_1(z)$ maybe not optimal even though the frequency response of $B(z)$ is optimal in the Chebyshev sense. From Eq.(2), we have

$$H_1(z) = z^{-2M}\{1 - \frac{B(z^2)H_0(z)}{z^{-2M}}\} = z^{-2M}\{1 - \hat{B}(z^2)\hat{H}_0(z)\}, \tag{20}$$

where

$$\hat{H}_0(z) = \frac{H_0(z)}{z^{-2N-1}} = \frac{1}{2}\{1 - \hat{A}(z^2)\}. \tag{21}$$

To force $H_1(z)$ to have an equiripple response in the band $[0, \omega_p]$, we have to optimalize the frequency response of $\hat{B}(z^2)\hat{H}_0(z)$ in the Chebyshev sense. We define an error function for $B(z)$ as

$$E_b(\omega) = \hat{H}_0(e^{j\frac{\omega}{2}})\hat{B}(e^{j\omega}) - 1. \tag{22}$$

Then, we can formulate $E_b(\omega)$ as shown in **3.2**, i.e.,

$$E_b(\omega_i) = \hat{H}_0(e^{j\frac{\omega_i}{2}})\hat{B}(e^{j\omega_i}) - 1 = \delta e^{j\theta(\omega_i)}, \tag{23}$$

where $\hat{H}_0(e^{j\frac{\omega}{2}})$ can be considered as a weighting function. Therefore, the design algorithm is the same as that of $A(z)$. In [3], there exists a bump of approximately $4dB$ at $\omega = \pi/2$, since $A(z)$ and $B(z)$ use the same allpass filter. In this paper, we can use general IIR filters $A(z)$ and $B(z)$, and appropriately choose the group delay $N$ and $M$ to suppress the bump around $\omega = \pi/2$. See design example in detail.

## 4. DESIGN EXAMPLE

We consider design of an IIR PR filter bank with $N = 7, M = 16, L_1 = L_3 = 10, L_2 = L_4 = 2$, and $\omega_p = 0.4\pi$. The filter bank is designed by using the proposed method. The obtained magnitude responses of $A(z)$ and $B(z)$ are shown in Fig.2, and the magnitude responses of $H_0(z)$ and $H_1(z)$ are shown in Fig.3. It can be seen in Fig.3 that both $H_0(z)$ and $H_1(z)$ have equiripple magnitude responses in the stopband, and $H_1(z)$ has not a bump at $\omega = \pi/2$. To obtain stable filters, $N$ and $M$ must be chosen as $N \geq 6$ and $M \geq 15$, i.e., $N_{min} = 6$ and $M_{min} = 15$. Therefore, the obtained $A(z)$ and $B(z)$ are guaranteed to be stable. The phase errors of $H_0(z)$ and $H_1(z)$ are shown in Fig.4, and it is clear that the phase errors are very small.

## 5. CONCLUSION

In this paper, we have proposed a new method for designing two channel biorthogonal IIR filter banks that satisfy both the perfect reconstruction and causal stable conditions. We have adopted the structurally perfect reconstruction implementation proposed in [3], and used general IIR filters to suppress the bump around $\omega = \pi/2$ caused when allpass filters are used in [3]. By using Remez multiple exchange algorithm, we have formulated the design problem of IIR PR filter banks in the form of a generalized eigenvalue problem. Therefore, the filter coefficients can be computed by solving the eigenvalue problem to get the positive minimum eigenvalue, and the optimal solution in the Chebyshev sense is easily obtained through a few iterations. Finally, we have designed one example to demonstrate the effectiveness of the proposed method.

## 6. REFERENCES


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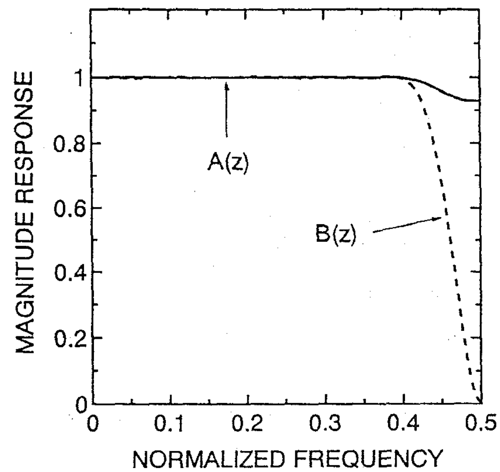


**Fig.2** Magnitude responses of $A(z)$ and $B(z)$.

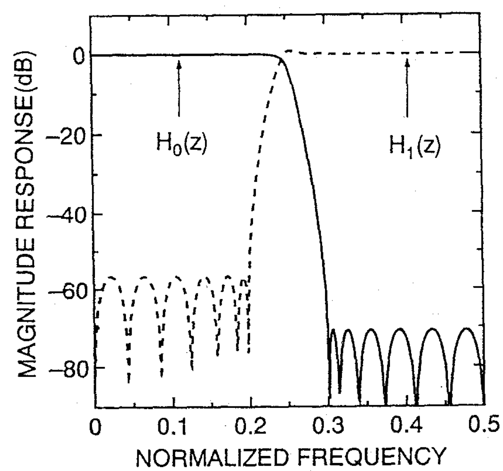


**Fig.3** Magnitude responses of $H_0(z)$ and $H_1(z)$.

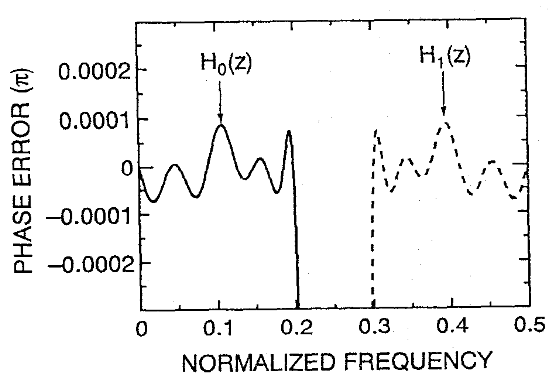


**Fig.4** Phase errors of $H_0(z)$ and $H_1(z)$.