DESIGN OF HILBERT TRANSFORM PAIRS OF ORTHONORMAL WAVELET BASES USING REMEZ EXCHANGE ALGORITHM

Xi ZHANG

Department of Information and Communication Engineering
The University of Electro-Communications
1–5–1 Chofugaoka, Chofu-shi, Tokyo, 182-8585 JAPAN
E-mail: xiz@ice.uec.ac.jp

ABSTRACT

This paper proposes a new method for designing Hilbert transform pairs of orthonormal wavelet bases proposed by Selesnick in [9]. The conventional method located as many zeros as possible at $z = -1$ to obtain the maximum number of vanishing moments. In this paper, we specify the number of zeros at $z = -1$, and then use the remaining degree of freedom to get the best possible frequency selectivity. The Remez exchange algorithm is applied in the stopband to approximate the equiripple magnitude response. Therefore, a set of filter coefficients can be obtained easily by solving a system of linear equations. Furthermore, the optimal solution is attained through a few iterations. Since the number of zeros at $z = -1$ can be specified arbitrarily, a new class of Hilbert transform pairs of orthonormal wavelet bases with the specified number of vanishing moments can be generated.

Keywords: Orthonormal wavelet basis, Hilbert transform pair, FIR filter, Vanishing moment, Remez exchange algorithm.

1. INTRODUCTION

Hilbert transform pairs of wavelet bases have been proposed and found to be successful in many applications of signal processing and image processing [5]∼[9], [12]. It has been proven in [8], [10] and [11] that the half-sample delay condition between two scaling lowpass filters is the necessary and sufficient condition for the corresponding wavelet bases to form a Hilbert transform pair. Several design procedures for the Hilbert transform pairs of wavelet bases have been presented in [5]∼[9] by using FIR filters, which are corresponding to the compactly supported wavelets. In [9], Selesnick has proposed a class of Hilbert transform pairs of orthonormal wavelet bases, where the corresponding scaling lowpass filters are constructed by using an allpass filter to meet the half-sample delay condition. This design method is simple and effective. The approximation accuracy of the half-sample delay is controlled only by the allpass filter. Thus, the design is for the scaling lowpass filters to satisfy the orthonormality condition and the regularity of wavelets. In [9], Selesnick has used the maximally flat allpass filter, and then located as many zeros of the scaling lowpass filters as possible at $z = -1$ to obtain the maximum number of vanishing moments, which results in the maximally flat magnitude response. It is well-known that frequency selectivity is a useful property for many applications of signal processing. However, the maximally flat filters have poor frequency selectivity [1]. For this reason, we will specify the number of zeros at $z = -1$, then use the remaining degree of freedom to get the best possible frequency selectivity.

In this paper, we propose a new design method for Hilbert transform pairs of orthonormal wavelet bases proposed by Selesnick in [9]. Since the scaling lowpass filters have satisfied the half-sample delay condition by using the allpass filter, only the orthonormality condition, regularity of wavelets, and frequency selectivity need to be considered. First, we locate the specified number of zeros at $z = -1$ from the viewpoint of regularity, and then derive a system of linear equations from the orthonormality condition. Next, we use the remaining degree of freedom to get the best possible frequency selectivity. We apply the Remez exchange algorithm in the stopband to obtain the equiripple magnitude response [2]. Therefore, a set of filter coefficients can be obtained easily by solving a system of linear equations, and the optimal solution is attained through a few iterations. Since the number of zeros at $z = -1$ can be specified arbitrarily, we can produce a new class of Hilbert transform pairs of orthonormal wavelet bases with the specified number of vanishing moments.

2. HILBERT TRANSFORM PAIRS OF WAVELET BASES

It is well-known that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z),\ G_i(z)\}$, where $i = 1, 2$. We assume that $H_i(z)$ are lowpass filter, and $G_i(z)$ are highpass. The orthonormality condition of $H_i(z)$ and $G_i(z)$ is given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0 \end{cases}. \tag{1}$$

Let $\phi_i(t), \psi_i(t)$ be the corresponding scaling and wavelet functions, respectively. The dilation and wavelet equations give the scaling and wavelet functions;

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n) \end{cases}, \tag{2}$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively.

It has been proven in [8], [10] and [11] that two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \tag{3}$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \tag{4}$$

if and only if two scaling lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{5}$$

where $\Psi_i(\omega)$ are the Fourier transform of $\psi_i(t)$. This is the so-called half-sample delay condition between two scaling lowpass filters. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. Eq.(5) is the necessary and sufficient condition for two orthonormal wavelet bases to form a Hilbert transform pair.

3. HILBERT TRANSFORM PAIRS OF ORTHONORMAL WAVELET BASES COMPOSED OF ALLPASS FILTER

It is known that the transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)}, \tag{6}$$

where

$$D(z) = 1 + \sum_{n=1}^{L} d(n)z^{-n}, \tag{7}$$

where L is the degree of $A(z)$ and $d(n)$ are real filter coefficients. Then, the phase response $\theta(\omega)$ of $A(z)$ is given by

$$\theta(\omega) = -L\omega + 2\tan^{-1}\frac{\sum_{n=1}^{L} d(n)\sin(n\omega)}{1 + \sum_{n=1}^{L} d(n)\cos(n\omega)}. \tag{8}$$

In [9], Selesnick has proposed that the scaling lowpass filters $H_1(z)$ and $H_2(z)$ have the following form;

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}) \end{cases}, \tag{9}$$

and $G_i(z) = -z^{-M}H_i(-z^{-1})$ for $i = 1, 2$, where M is the degree of $H_i(z)$ and is an odd number.

Since $H_1(z)$ and $H_2(z)$ have the common divisor $F(z)$, we have

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z). \tag{10}$$

Therefore, if $A(z)$ in Eq.(6) is an approximate half-sample delay;

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{11}$$

then the half-sample delay condition in Eq.(5) is achieved approximately. Thus, two wavelet bases form an approximate Hilbert transform pair.

There exist many design methods for allpass filters to approximate a fractional delay response, for example, the maximally flat, least squares [3], equiripple approximations [4], and so on. In [9], the maximally flat fractional delay allpass filter was adapted, and $\omega = 0$ was chosen for the point of approximation. However, the approximation error will increase as ω goes away from the point of approximation in the maximally flat approximation. Thus, it will be better if the minimax (Chebyshev) phase approximation of allpass filters is used, e.g., [4].

Once $A(z)$ is determined, we need to design $F(z)$ for $H_1(z)$ and $H_2(z)$. To obtain wavelet bases with K vanishing moments, we have

$$F(z) = Q(z)(1 + z^{-1})^K. \tag{12}$$

Thus

$$\begin{cases} H_1(z) = Q(z)(1 + z^{-1})^K D(z) \\ H_2(z) = Q(z)(1 + z^{-1})^K z^{-L}D(z^{-1}) \end{cases}. \tag{13}$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(1 + z)^K(1 + z^{-1})^K D(z)D(z^{-1}) \end{aligned}. \tag{14}$$

Defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-N}^{N} r(n)z^{-n}, \tag{15}$$

$$S(z) = (z + 2 + z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \tag{16}$$

where $r(n) = r(-n)$ for $1 \leq n \leq N$ and $s(n) = s(-n)$ for $1 \leq n \leq L + K$, we can write the orthonormality condition in Eq.(1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n - k)r(k) = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases}, \tag{17}$$

where $I_{min} = \max\{-N, 2n - L - K\}$ and $I_{max} = \min\{N, 2n + L + K\}$. Note that $P(z)$ is a halfband filter, thus $N + L + K = M$ is an odd number, where the degree of $H_i(z)$ is $M = N + L + K$. In Eq.(17), there are $(M + 1)/2$ equations with respect to $(N + 1)$ unknown coefficients $r(n)$. Therefore, it is clear that we can obtain the only solution if $(M + 1)/2 = N + 1$, which is corresponding to the maximal K ($K_{max} = N - L + 1 = (M + 1)/2 - L$ for given N and L), having the maximum number of vanishing moments. In [9], Selesnick had chosen $N = L + K - 1$ and obtained the filter of minimal degree for given L and K. Both are equivalent and have the maximally flat magnitude response. However, it is known in [1] that the maximally flat filters have poor frequency selectivity. In the following, we will consider how to design $R(z)$ with an improved frequency selectivity.

4. HILBERT TRANSFORM PAIRS OF WAVELET BASES WITH IMPROVED FREQUENCY SELECTIVITY

It is well-known that frequency selectivity is a useful property for many applications of signal processing. In this section, we firstly specify the number of zeros of $H_i(z)$ at $z = -1$ from the viewpoint of regularity, and then use the remaining degree of freedom to get the best possible frequency selectivity. We consider the case of $K < (M + 1)/2 - L$. Besides satisfying the orthonormality condition

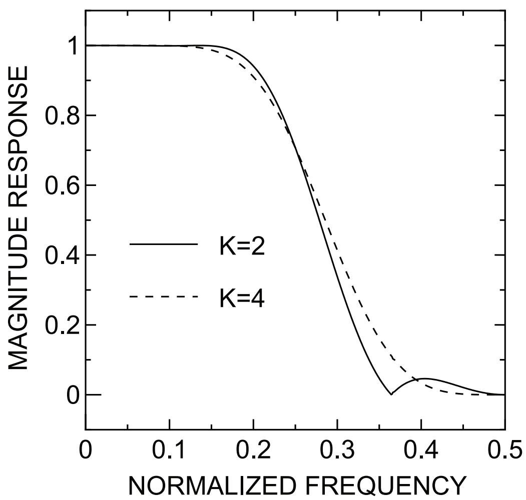

Fig. 1. Magnitude responses of scaling lowpass filters $H_i(z)$.

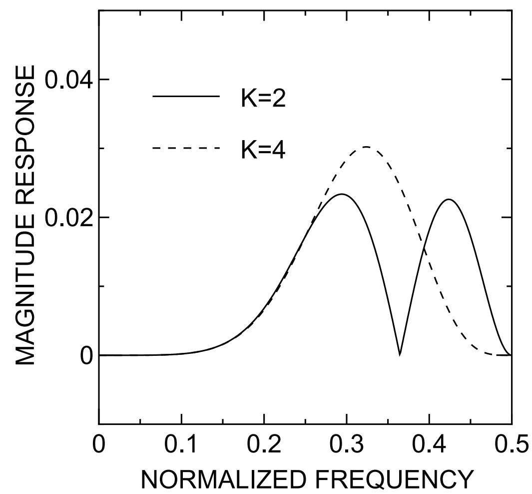

Fig. 2. Magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$.

in Eq.(17), we want to obtain an equiripple magnitude response of $P(z)$ in the stopband by using the remaining degree of freedom. The remaining degree of freedom is $(M+1)/2 - L - K$. Since zeros on the unit circle except $z = \pm 1$ are complex-conjugate pair, $(M+1)/2 - L - K$ should be even, i.e., $(M+1)/2 - L - K = 2m$.

From Eqs.(14), (15) and (16), we have

$$P(e^{j\omega}) = R(e^{j\omega})S(e^{j\omega}), \tag{18}$$

where

$$\begin{cases} R(e^{j\omega}) = r(0) + 2\sum_{n=1}^{N} r(n)\cos(n\omega) \\ S(e^{j\omega}) = s(0) + 2\sum_{n=1}^{L+K} s(n)\cos(n\omega) \end{cases}. \tag{19}$$

We apply the Remez exchange algorithm in the stopband $[\omega_s, \pi]$, where ω_s $(\pi/2 < \omega_s < \pi)$ is the cut-off frequency of the stopband of $H_i(z)$. Let $\omega_s = \omega_0 < \omega_1 < \cdots < \omega_{2m} < \pi$ be the extremal frequencies in the stopband, we formulate $P(e^{j\omega})$ as

$$P(e^{j\omega_i}) = R(e^{j\omega_i})S(e^{j\omega_i}) = (1 + (-1)^i)\delta, \tag{20}$$

where $\delta > 0$ is an error. Note that we force $P(e^{j\omega}) \geq 0$ to permit spectral factorization of $R(z)$. We rewrite Eq.(20) as

$$R(e^{j\omega_i}) = \frac{1 + (-1)^i}{S(e^{j\omega_i})}\delta \qquad (i = 0, 1, \cdots, 2m). \tag{21}$$

By substituting Eq.(19) into Eq.(21), we have

$$r(0) + 2\sum_{n=1}^{N} r(n)\cos(n\omega_i) - \frac{1 + (-1)^i}{S(e^{j\omega_i})}\delta = 0, \tag{22}$$

for $i = 0, 1, \cdots, 2m$. It should noted that Eqs.(17) and (22) have $(M+1)/2 + 2m + 1 = N + 2$ equations with respect to $(N+1)$ filter coefficients $r(n)$ plus one error δ. Therefore, we can solve the system of linear equations in Eqs.(17) and (22) to obtain a set of filter coefficients $r(n)$. Furthermore, we make use of an iteration procedure to obtain the equiripple magnitude response of $P(z)$. Thus the optimal filter coefficients $r(n)$ can be easily obtained through a few iterations. Similar to the way proposed in [9], we can obtain $Q(z)$ from $R(z)$ by using a spectral factorization approach, where $Q(z)$ is not unique. The design algorithm is shown as follows.

5. DESIGN ALGORITHM

Procedure {Design of Hilbert Transform Pairs of Orthonormal Wavelet Bases with Improved Frequency Selectivity}

Begin

1) Read N, K, L, and ω_s.
2) Design $A(z)$ to get $d(n)$, and use Eq.(16) to compute $s(n)$.
3) Select initial extremal frequencies Ω_i $(\omega_s = \Omega_0 < \Omega_1 < \cdots < \Omega_{2m} < \pi)$ equally spaced in the stopband.

Repeat

4) Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, 2m$.
5) Solve Eqs.(17) and (22) to obtain a set of filter coefficients $r(n)$.
6) Search the peak frequencies Ω_i $(\omega_s = \Omega_0 < \Omega_1 < \cdots < \Omega_{2m} < \pi)$ of $P(e^{j\omega})$ in the stopband.

Until

Satisfy the following condition for a prescribed small constant ϵ (e.g., $\epsilon = 10^{-8}$);

$$\sum_{i=1}^{2m} |\omega_i - \Omega_i| < \epsilon$$

7) Factorize $R(z)$ to get $Q(z)$, and use Eq.(13) to construct $H_1(z)$ and $H_2(z)$.

End.

6. DESIGN EXAMPLE

In this section, we present one example to demonstrate the effectiveness of the design method proposed in this paper. First, we have designed a Hilbert transform pair of orthonormal wavelet bases with the maximum number of vanishing moments as proposed in [9]. The magnitude response of the scaling lowpass filters $H_i(z)$ with $M =$

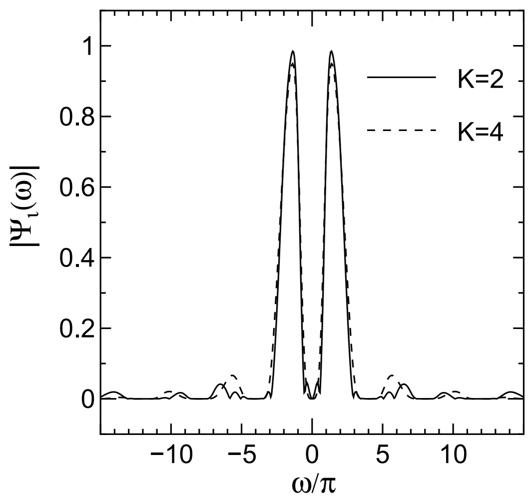

Fig. 3. Magnitude responses of $\Psi_i(\omega)$.

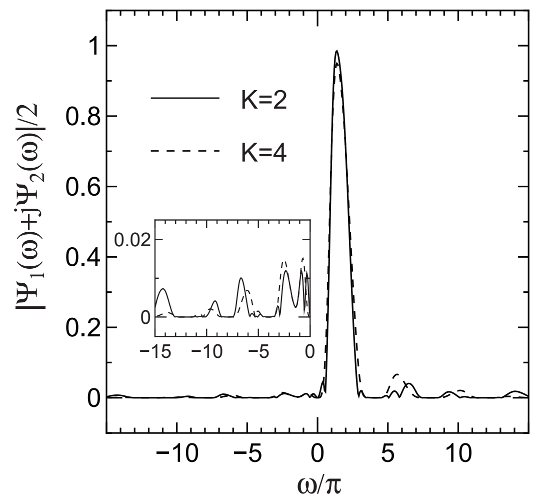

Fig. 4. Magnitude responses of $(\Psi_1(\omega) + j\Psi_2(\omega))/2$.

11 is shown in dash line in Fig.1, where $N = 5, K = 4, L = 2$. Next, we considered the design of $H_i(z)$ with the same degree, i.e., $M = 11$. We set $K = 2$ and $N = 7$, thus the remaining degree of freedom is $2m = (M + 1)/2 - L - K = 2$. The stopband of $H_i(z)$ is set to $[\omega_s = 0.7\pi, \pi]$. We have obtained the equiripple magnitude response of $P(z)$ by using the design algorithm proposed in the preceding section. The resulting magnitude response of $H_i(z)$ is shown also in solid line in Fig.1. It is seen in Fig.1 that it is more sharp than the maximally flat filter. The magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ are shown in Fig.2. It is clear in Fig.2 that the maximum error of the proposed filter pair is smaller than one proposed in [9]. Moreover, the spectrum $\Psi_i(\omega)$ of the obtained wavelet functions $\psi_i(t)$ are shown in Fig.3, while the spectrum $\Psi_1(\omega) + j\Psi_2(\omega)$ of $\psi_1(t) + j\psi_2(t)$ in Fig.4, which approximate zero for $\omega < 0$.

7. CONCLUSION

In this paper, we have proposed a new design method for Hilbert transform pairs of orthonormal wavelet bases. First, we have located the specified number of zeros at $z = -1$ for the scaling lowpass filters, and then derived a system of linear equations from the orthonormality condition. Next, we have applied the Remez exchange algorithm in the stopband to obtain the equiripple magnitude response. Therefore, a set of filter coefficients can be obtained easily by solving a system of linear equations, and the optimal solution is attained through a few iterations. The proposed design procedure is computationally efficient because it retains the speed inherent in the Remez exchange algorithm. Since the number of zeros at $z = -1$ can be specified arbitrarily, a new class of Hilbert transform pairs of orthonormal wavelet bases with the specified number of vanishing moments can be generated.

8. REFERENCES

[1] S. K. Mitra and J. F. Kaiser, "Handbook for Digital Signal Processing," Wiley, New York, 1993.

[2] T. W. Parks and J. H. McClellan, "Chebyshev approximation for nonrecursive digital filters with linear phase," IEEE Trans. Circuit Theory, vol.CT-19, no.2, pp.189–194, Mar. 1972.

[3] T. I. Laakso, V. Valimaki, M. Karjalainen and U. K. Laine, "Splitting the unit delay: Tools for fractional delay filter design," IEEE Signal Processing Mag., vol.13, no.1, pp.30–60, Jan. 1996.

[4] X. Zhang and H. Iwakura, "Design of IIR digital allpass filters based on eigenvalue problem," IEEE Trans. Signal Processing, vol.47, no.2, pp.554–559, Feb. 1999.

[5] N. G. Kingsbury, "The dual-tree complex wavelet transform: A new technique for shift invariance and directional filters," in Proc. 8th IEEE DSP Woekshop, Utan, no.86, Aug. 1998.

[6] N. G. Kingsbury, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in Proc. IEEE ICIP, Vancouver, Canada, vol.2, pp.375–378, Sep. 2000.

[7] N. G. Kingsbury, "Complex wavelets for shift invariant analysis and filtering of signals," Appl. Comput. Harmon. Anal., vol.10, no.3, pp.234–253, May 2001.

[8] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," IEEE Signal Processing Letters, vol.8, no.6, pp.170–173, Jun. 2001.

[9] I. W. Selesnick, "The design of approximate Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.50, no.5, pp.1144–1152, May 2002.

[10] H. Ozkaramanli and R. Yu, "On the phase condition and its solution for Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.51, no.12, pp.3293–3294, Dec. 2003.

[11] R. Yu and H. Ozkaramanli, "Hilbert transform pairs of orthogonal wavelet bases: Necessary and sufficient conditions," IEEE Trans. Signal Processing, vol.53, no.12, pp.4723–4725, Dec. 2003.

[12] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The dual-tree complex wavelet transform," IEEE Signal Processing Mag., vol.22, no.6, pp.123–151, Nov. 2005.