DESIGN OF Q-SHIFT FILTERS WITH IMPROVED VANISHING MOMENTS FOR DTCWT

Xi ZHANG

Department of Communication Engineering and Informatics
The University of Electro-Communications
Chofu-shi, Tokyo, 182-8585 Japan
E-mail: zhangxi@uec.ac.jp

ABSTRACT

Q-shift filters have been used in the Dual Tree Complex Wavelet Transform (DTCWT), and are required to have a well-controlled group delay. This paper proposes a new method for designing Q-shift filters with improved vanishing moments. The proposed design method allows the flatness condition of the group delay response to be included along with the vanishing moments and orthogonality of wavelets. Therefore, the resulting Q-shift filters possess flat group delay responses. Moreover, the number of vanishing moments can be arbitrarily specified by locating the desired number of zeros at $z = -1$. Finally, one example is presented to demonstrate the effectiveness of the proposed design method.

Keywords: DTCWT, Orthogonal wavelet, Hilbert transform pair, Q-shift filter, FIR filter, Flat frequency response, Vanishing moment.

1. INTRODUCTION

The Dual Tree Complex Wavelet Transform (DTCWT) was originally proposed by Kingsbury in [2], and has been found to be successful in many applications of signal processing and image processing [2]∼[10]. DTCWT has the following significant properties over DWT (Discrete Wavelet Transform): approximate shift invariance, and good directional selectivity for multidimensional signals. It has been shown in [5] that two scaling lowpass filters are required to satisfy the half-sample delay condition, thus the corresponding wavelet bases form a Hilbert transform pair.

Several design procedures for DTCWT had been presented in [2]∼[7]. In [6], Selesnick had proposed a design technique using the maximally flat allpass filters. This method is simple and effective, but the resulting filters have a non-linear phase response. In [3] and [4], Kingsbury introduced Q-shift filters in order to provide the improved orthogonality and symmetry properties. Q-shift filters are required to have a linear phase response. The design technique proposed in [3] and [4] was based on the optimization of a set of rotations θ_i in the polyphase structure, but this is a highly non-linear problem and only works well for relatively short filters. In [7], Kingsbury had proposed an alternative technique for optimizing Q-shift filters, which works effectively for filters of lengths up to 50 or more taps. This method was based on the minimization of energy of $H_{L2}(z)$ in $[\frac{\pi}{3}, \pi]$, instead of the direct approximation of group delay for $H_0(z)$.

In this paper, we propose a new method for designing Q-shift filters with improved vanishing moments. Differently from the technique proposed in [7], we consider the direct approximation of group delay response for Q-shift filters. We specify the degree of flatness for the group delay response at $\omega = 0$, and then derive a set of linear equations from this flatness condition. Moreover, we locate the specified number of zeros at $z = -1$ from the viewpoint of vanishing moment. Therefore, the resulting Q-shift filters have a flat group delay response and the specified number of vanishing moments. In the proposed method, the filter coefficients can be obtained easily by iteratively solving a set of linear equations only. Finally, one example is presented to demonstrate the effectiveness of the proposed design method.

2. Q-SHIFT FILTERS FOR DTCWT

It is known that DTCWT employs two real DWTs; the first DWT gives the real part of DTCWT and the second DWT is the imaginary part. The second wavelet basis is required to be the Hilbert transform of the first wavelet basis.

Let $\phi_H(t), \phi_G(t)$ and $\psi_H(t), \psi_G(t)$ be the scaling and wavelet functions of two DWTs, respectively. It has been proven in [5], [8] and [9] that two wavelet functions $\psi_H(t)$ and $\psi_G(t)$ form a Hilbert transform pair;

$$\psi_G(t) = \mathcal{H}\{\psi_H(t)\}, \tag{1}$$

that is

$$\Psi_G(\omega) = \begin{cases} -j\Psi_H(\omega) & (\omega > 0) \\ j\Psi_H(\omega) & (\omega < 0) \end{cases}, \tag{2}$$

if and only if two scaling lowpass filters satisfy

$$G(e^{j\omega}) = H(e^{j\omega})e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{3}$$

where $\Psi_H(\omega), \Psi_G(\omega)$ are the Fourier transform of $\psi_H(t), \psi_G(t)$, respectively. This is the so-called half-sample delay condition between two scaling lowpass filters $H(z)$ and $G(z)$, which has been generalized in [11]. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. Eq.(3) is the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair [9].

In [3] and [4], Kingsbury had proposed Q-shift filters in order to provide the improved orthogonality and symmetry properties. One scaling lowpass filter is chosen to be the time reverse of another filter;

$$G(z) = z^{-N}H(z^{-1}), \tag{4}$$

where $H(z)$ is FIR filter of degree N. Its transfer function is given by

$$H(z) = \sum_{n=0}^{N} h(n)z^{-n}, \tag{5}$$

where $h(n)$ are real filter coefficients and N is an odd number.

Q-shift filters are required to have a linear phase response. That is, the desired phase response of $H(z)$ is

$$\theta_d(\omega) = -(\frac{N}{2} - \frac{1}{4})\omega. \tag{6}$$

Therefore, the phase response of $G(z)$ will be $-(\frac{N}{2}+\frac{1}{4})\omega$, thus two scaling lowpass filters safisfy the half-sample delay condition.

3. DESIGN OF Q-SHIFT FILTERS

In this section, we discuss the design of Q-shift filters. From Eq.(5), the phase response of $H(z)$ is given by

$$\theta(\omega) = -\tan^{-1}\frac{\sum_{n=0}^{N} h(n)\sin(n\omega)}{\sum_{n=0}^{N} h(n)\cos(n\omega)}. \tag{7}$$

Thus, the difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d(\omega)$ is

$$\theta_e(\omega) = \theta(\omega) - \theta_d(\omega) = 2\tan^{-1}\frac{N(\omega)}{D(\omega)}, \tag{8}$$

where

$$\begin{cases} N(\omega) = \sum_{n=0}^{N} h(n)\sin((\frac{N}{2} - n - \frac{1}{4})\omega) \\ D(\omega) = \sum_{n=0}^{N} h(n)\cos((\frac{N}{2} - n - \frac{1}{4})\omega) \end{cases}. \tag{9}$$

There are many criterions in the group delay approximation, e.g., maximally flat, least square, equiripple approximation, and so on [1]. In this paper, we consider the maximally flat approximation. $H(z)$ is required to have the specified degree of flatness at $\omega = 0$ for the group delay response, that is,

$$\begin{cases} \tau(0) = \frac{N}{2} - \frac{1}{4} \\ \left.\frac{\partial^{2r}\tau(\omega)}{\partial\omega^{2r}}\right|_{\omega=0} = 0 \quad (r = 1, 2, \cdots, L-1) \end{cases}, \tag{10}$$

where $L\,(>0)$ is a parameter that controls the degree of flatness. Since $\tau(\omega) = -\frac{\partial\theta(\omega)}{\partial\omega}$, Eq.(10) is equivalent to

$$\left.\frac{\partial^{2r+1}\theta_e(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0 \qquad (r = 0, 1, \cdots, L-1). \tag{11}$$

By using Eq.(8), Eq.(11) can be reduced to

$$\left.\frac{\partial^{2r+1}N(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0 \qquad (r = 0, 1, \cdots, L-1). \tag{12}$$

By substituting $N(\omega)$ in Eq.(9) into Eq.(12), we can derive a set of linear equations as follows;

$$\sum_{n=0}^{N}(\frac{N}{2} - n - \frac{1}{4})^{2r+1}h(n) = 0 \quad (r = 0, 1, \cdots, L-1). \tag{13}$$

It is clear that there are L equations in Eq.(13) with respect to $(N+1)$ unknown coefficients $h(n)$.

In addition to the phase condition given in Eq.(13), $H(z)$ is required to satisfy the condition of orthonormality and to have the maximum number of vanishing moments.

To obtain orthonormal wavelet bases, $H(z)$ must satisfy

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2. \tag{14}$$

We rewrite the condition of orthonormality in Eq.(14) as

$$\sum_{k=0}^{N-2n} h(k+2n)h(k) = \delta(n) = \begin{cases} 1 & (n=0) \\ 0 & (n>0) \end{cases}, \tag{15}$$

where there are $(N+1)/2$ equations with respect to $h(n)$.

Moreover, $H(z)$ must have K zeros at $z = -1$ to get the desired number of vanishing moments;

$$H(z) = Q(z)(1+z^{-1})^K. \tag{16}$$

Therefore, we have

$$\left.\frac{\partial^r H(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\pi} = 0 \qquad (r = 0, 1, \cdots, K-1). \tag{17}$$

By substituting $H(e^{j\omega})$ in Eq.(5) into Eq.(17), we derive a set of linear equations as follows;

$$\sum_{n=0}^{N}(-1)^n n^r h(n) = 0 \qquad (r = 0, 1, \cdots, K-1), \tag{18}$$

where there are K equations with respect to $h(n)$. If $K + L = (N+1)/2$, then there are $K + L + (N+1)/2 = N+1$ equations in Eq.(13), Eq.(15) and Eq.(18) with respect to $(N+1)$ unknowns $h(n)$. Therefore, we can obtain the filter coefficients $h(n)$ by solving Eq.(13), Eq.(15) and Eq.(18).

For example, we discuss the case of $N = 3$. Since $K + L = (N+1)/2 = 2$ and $K > 0, L > 0$, we can choose $K = L = 1$ only. Thus, from Eq.(15), Eq.(18) and Eq.(13), we have

$$\begin{cases} h(0)^2 + h(1)^2 + h(2)^2 + h(3)^2 = 1 \\ h(0)h(2) + h(1)h(3) = 0 \\ h(0) - h(1) + h(2) - h(3) = 0 \\ \frac{5}{4}h(0) + \frac{1}{4}h(1) - \frac{3}{4}h(2) - \frac{7}{4}h(3) = 0 \end{cases}. \tag{19}$$

It is clear that there are four solutions in Eq.(19). If $h(n)$ is one solution, then $-h(n)$ is a solution too. Assuming $H(1) > 0$, thus, we have two solutions;

$$\begin{aligned} H_1(z) = {} & 0.0996151398 + 0.7842683367z^{-1} \\ & + 0.6074916414z^{-2} - 0.0771615555z^{-3}, \end{aligned}$$

$$\begin{aligned} H_2(z) = {} & 0.7842683367 + 0.0996151398z^{-1} \\ & - 0.0771615555z^{-2} + 0.6074916414z^{-3}. \end{aligned}$$

The magnitude responses of two filters $H_1(z)$ and $H_2(z)$ are shown in Fig.1, and the group delay responses are shown in Fig.2 also. It is clear that $H_1(z)$ and $H_2(z)$ have different frequency responses, regardless both satisfy the condition in Eq.(19). $H_1(z)$ has a better magnitude response and more flat group delay response than $H_2(z)$. Therefore, we should choose $H_1(z)$ as the optimal solution. The rotations are $\theta = \{1.8391, -0.8391\}\pi/4$ in the polyphase structure.

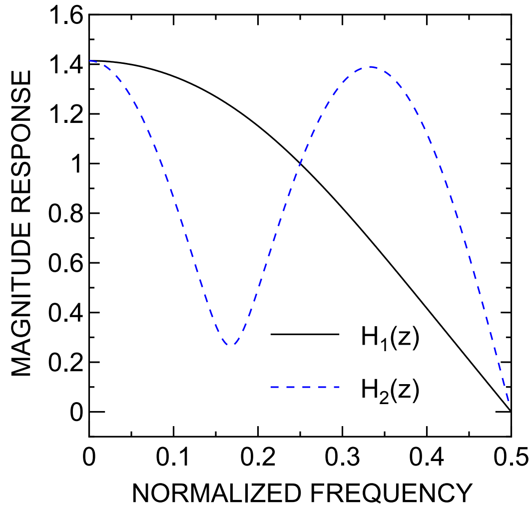

Fig. 1. Magnitude responses of Q-shift filters with $N = 3$.

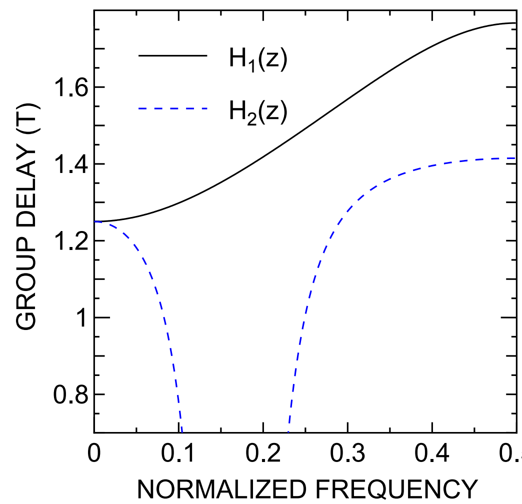

Fig. 2. Group delay responses of Q-shift filters with $N = 3$.

4. AN ITERATIVE PROCEDURE FOR Q-SHIFT FILTERS

It is difficult to solve the non-linear problem in Eq.(13), Eq.(15) and Eq.(18), particularly if the filter degree N is large, because Eq.(15) is a set of quadratic constraints on the filter coefficients $h(n)$. In this section, we linearize the problem and use an iterative procedure to compute a set of filter coefficients $h(n)$, similarly to the technique proposed by Kingsbury in [7].

Let $h^{(i)}(n)$ be the filter coefficients at ith iteration, and is given by

$$h^{(i)}(n) = h^{(i-1)}(n) + \Delta h^{(i)}(n). \tag{20}$$

Therefore, Eq.(15) becomes

$$\sum_{k=0}^{N-2n} [h^{(i-1)}(k+2n)h^{(i-1)}(k) + h^{(i-1)}(k+2n)\Delta h^{(i)}(k) + h^{(i-1)}(k)\Delta h^{(i)}(k+2n) + \Delta h^{(i)}(k)\Delta h^{(i)}(k+2n)] = \delta(n) \tag{21}$$

If $\Delta h^{(i)}(k)$ is assumed to become small as i increases, the term $\Delta h^{(i)}(k)\Delta h^{(i)}(k+2n)$ can be neglected. Thus we have

$$\sum_{k=0}^{N}[h^{(i-1)}(k+2n) + h^{(i-1)}(k-2n)]\Delta h^{(i)}(k) = \delta(n) - \sum_{k=0}^{N-2n} h^{(i-1)}(k+2n)h^{(i-1)}(k) \tag{22}$$

where $h^{(i-1)}(k) = 0$ for $k < 0,\ k > N$. Moreover, Eq.(13) and Eq.(18) become

$$\sum_{n=0}^{N}(\frac{N}{2} - n - \frac{1}{4})^{2r+1}\Delta h^{(i)}(n) = \sum_{n=0}^{N}(n + \frac{1}{4} - \frac{N}{2})^{2r+1}h^{(i-1)}(n) \tag{23}$$

$$\sum_{n=0}^{N}(-1)^n n^r \Delta h^{(i)}(n) = \sum_{n=0}^{N}(-1)^{(n+1)}n^r h^{(i-1)}(n). \tag{24}$$

Therefore, we can get $\Delta h^{(i)}(n)$ by solving the set of linear equations in Eq.(22), Eq.(23) and Eq.(24), if $h^{(i-1)}(n)$ are known. The filter coefficients $h^{(i)}(n)$ are updated by $\Delta h^{(i)}(n)$ in Eq.(20).

To converge to the optimal solution, a set of good initial coefficients $h^{(0)}(n)$ are needed. It is known that $P(z) = H(z)H(z^{-1})$ is a linear phase half-band filter. We firstly design $P(z)$ as the maximally flat half-band filter, and choose the magnitude response of $H(z)$ as $|H(e^{j\omega})| = |P(e^{j\omega})|^{\frac{1}{2}}$. We set its phase response as $-(k+\frac{1}{4})\omega$. That is, $H(e^{j\omega}) = |P(e^{j\omega})|^{\frac{1}{2}}e^{-j(k+\frac{1}{4})\omega}$. Then, a set of initial coefficients $h^{(0)}(n)$ are computed by taking $(N+1)$-point IDFT. As mentioned in the preceding section, there are more than one solutions. We choose k from 0 to N to get different initial coefficients $h^{(0)}(n)$. Thus, we can obtain more than one solutions and choose the optimal filter coefficients from these solutions.

5. DESIGN EXAMPLE

In this section, we present one example to demonstrate the effectiveness of the proposed design method. We have designed Q-shift filters with $N = 9$ by using the proposed method. Firstly, $K = 4, L = 1$ was chosen, and the resulting rotations are $\theta = \{-0.5717, 1.6618, 1.5555, -1.3458, -0.2998\}\pi/4$. Its magnitude and group delay responses are shown in solid line in Fig.3 and Fig.4, respectively. For comparison, the magnitude and group delay responses of the filters with $K = 1, L = 4$ ($\theta = \{-1.5404, -1.7338, -1.6680, 1.9822, -0.0400\}\pi/4$), $K = 2, L = 3$ ($\theta = \{-1.5356, 0.7675, -1.9492, -0.4892, 0.2066\}\pi/4$), and designed by Kingsbury are shown also. It is seen in Fig.3 that the magnitude response becomes more sharp as K increases. It is noted that Q-shift filter designed by Kingsbury has a sharp magnitude response, but only one zero at $z = -1$, which means only one vanishing moment. It is clear in Fig.4 that the group delay responses become more flat as L increases, and it is better than the filter designed by Kingsbury.

It is clear in Eq.(3) that $G(e^{j\omega})$ needs to be approximated to $H(e^{j\omega})e^{-j\frac{\omega}{2}}$. For the purpose of comparison, we define the error function $E(\omega)$ as

$$E(\omega) = G(e^{j\omega}) - H(e^{j\omega})e^{-j\frac{\omega}{2}}. \tag{25}$$

The magnitude responses of $E(\omega)$ of four Q-shift filters are shown in Fig.5. It is seen in Fig.5 that Q-shift filter with $K = 1, L = 4$ has the minimum error. This means that it has an improved analyticity.

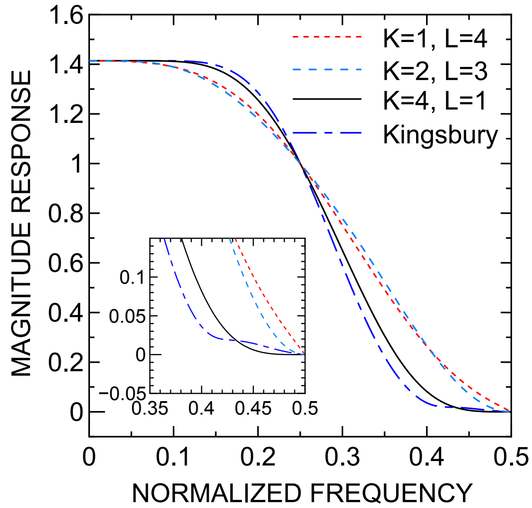

Fig. 3. Magnitude responses of Q-shift filters with $N = 9$.

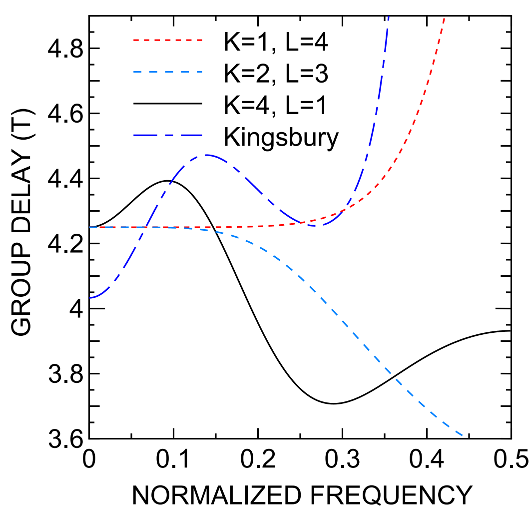

Fig. 4. Group delay responses of Q-shift filters with $N = 9$.

6. CONCLUSION

In this paper, we have proposed a new iterative procedure for designing Q-shift filters with improved vanishing moments in DTCWT. We have firstly discussed the direct approximation of group delay response for Q-shift filters. The degree of flatness can be specified arbitrarily at $\omega = 0$ for the group delay response, then a set of linear equations have been derived from this flatness condition. Moreover, we have located the specified number of zeros at $z = -1$ to obtain the vanishing moment. Therefore, the resulting Q-shift filters have a flat group delay response and the specified number of vanishing moments. The filter coefficients can be computed easily by iteratively solving a set of linear equations only. Therefore, the proposed design method is computationally efficient. Finally, one example is presented to demonstrate the effectiveness of the design method proposed in this paper.

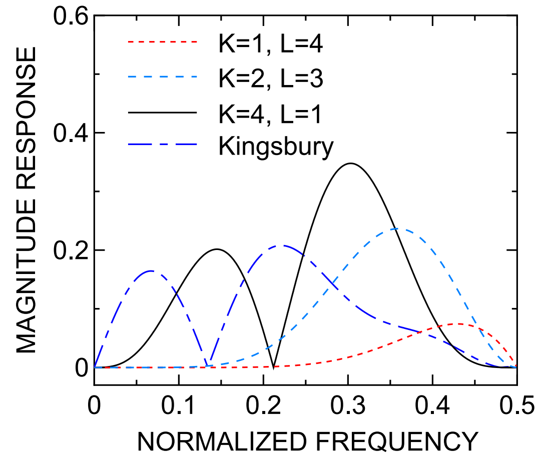

Fig. 5. Magnitude responses of $E(\omega)$.

7. REFERENCES

[1] T. I. Laakso, V. Valimaki, M. Karjalainen and U. K. Laine, "Splitting the unit delay: Tools for fractional delay filter design," IEEE Signal Processing Mag., vol.13, no.1, pp.30–60, Jan. 1996.

[2] N. G. Kingsbury, "The dual-tree complex wavelet transform: A new technique for shift invariance and directional filters," in Proc. 8th IEEE DSP Workshop, Utan, no.86, Aug. 1998.

[3] N. G. Kingsbury, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in Proc. ICIP2000, Vancouver, Canada, vol.2, pp.375–378, Sep. 2000.

[4] N. G. Kingsbury, "Complex wavelets for shift invariant analysis and filtering of signals," Appl. Comput. Harmon. Anal., vol.10, no.3, pp.234–253, May 2001.

[5] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," IEEE Signal Processing Letters, vol.8, no.6, pp.170–173, Jun. 2001.

[6] I. W. Selesnick, "The design of approximate Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.50, no.5, pp.1144–1152, May 2002.

[7] N. G. Kingsbury, "Design of Q-shift complex wavelets for image processing using frequency domain energy minimization," in Proc. ICIP2003, Barcelona, Spain, vol.1, pp.1013–1016, Sep. 2003.

[8] H. Ozkaramanli and R. Yu, "On the phase condition and its solution for Hilbert transform pairs of wavelet bases," IEEE Trans. Signal Processing, vol.51, no.12, pp.3293–3294, Dec. 2003.

[9] R. Yu and H. Ozkaramanli, "Hilbert transform pairs of orthogonal wavelet bases: Necessary and sufficient conditions," IEEE Trans. Signal Processing, vol.53, no.12, pp.4723–4725, Dec. 2003.

[10] I. W. Selesnick, R. G. Baraniuk and N. G. Kingsbury, "The dual-tree complex wavelet transform," IEEE Signal Processing Mag., vol.22, no.6, pp.123–151, Nov. 2005.

[11] C. Chaux, L. Duval and J. -C. Pesquet, "Image analysis using a dual-tree M-band wavelet transform," IEEE Trans. Image Processing, vol.15, no.8, pp.2397–2412, Aug. 2006.