A NOVEL DESIGN OF BIORTHOGONAL GRAPH WAVELET FILTER BANKS

Xi Zhang

Department of Communication Engineering and Informatics
The University of Electro-Communications
Chofu-shi, Tokyo, 182-8585 Japan
E-mail: zhangxi@uec.ac.jp

ABSTRACT

This paper proposes a novel method for designing compactly supported biorthogonal graph wavelet filter banks with flat spectral responses. We firstly construct a class of biorthogonal graph filter banks by using the polynomial half-band kernels, and then present a design method for the polynomial half-band kernel. The proposed design method utilizes the PBP (Parametric Bernstein Polynomial), which ensures that the half-band kernel has the specified zeros at $\lambda = 2$, that is, the flatness constraints both in passband and stopband. Therefore, the resulting graph filters have the flat spectral responses. Furthermore, we apply the Remez exchange algorithm to minimize the spectral error by using the remaining degree of freedom. Finally, two design examples are given to demonstrate the effectiveness of the design method proposed in this paper.

***Index Terms*—** Graph signal processing, graph wavelets, biorthogonal filter bank, polynomial half-band kernel.

1. INTRODUCTION

Signal processing on graphs has been found to be useful in numerous applications such as biological, energy, social, sensor, and transportation networks [7], [8]. Signal processing on graphs aims to extend the classical signal processing concepts and methodologies to signals defined on general graphs. Major challenges are how to efficiently analyze, compress and process large amounts of signals. It is well-known in [1]$\sim$[3] that wavelet filter banks can provide a sparse representation of signals as a widely used signal processing tool. Recently, there are many works to extend the classical wavelet transforms to signals on graphs, namely, graph wavelet transforms [9]$\sim$[20]. However, a drawback is that those transforms proposed in [9]$\sim$[11] and [14] are not critically sampled. Critical sampling is important for compact representation of signals, e.g., compression. Critically sampled graph wavelet filter banks have been proposed in [12], [13], [15], [17], [19] and [20] also. Furthermore, the filters are required to be compactly supported in the graph, i.e., the output at each vertex is computed exactly from the signal at that vertex and its K-hop neighborhood. It can be achieved by a polynomial approximation of the desired spectral kernel. The lifting-based graph wavelet filter banks in [12], [13] are compactly supported, but not orthogonal. Two channel orthogonal graph wavelet filter bank *graph-QMF* has been proposed in [15], [19] and [20], but the perfect reconstruction and orthogonality cannot be exactly achieved by using a polynomial approximation of the kernel filters. Narang and Ortega have proposed a simple design technique based on the Meyer's wavelet construction to obtain near orthogonal graph wavelet filter banks in [15]. However, The reconstruction error cannot be directly controlled and may be quite large. In [19], Tay and Lin have proposed a constrained optimization method to minimize the reconstruction error, which uses the PBP for generating the initial solution. In [20], *graph-QMF* with flatness constraints has been discussed, where only one extra parameter is used to reduce the reconstruction error. In [17], Narang and Ortega have also proposed a class of compactly supported biorthogonal graph wavelet filter banks *graphBior* by relaxing the condition of orthogonality, and given a design method based on the Cohen-Daubechies-Feauveau's wavelet construction of factorizing the maximally flat half-band filter. However, the resulting spectral responses are poor.

In this paper, we propose a novel design method of critically sampled compactly supported biorthogonal graph wavelet filter banks with flat spectral responses. Firstly, we use the polynomial half-band kernels to construct a class of biorthogonal graph filter banks, where the perfect reconstruction condition is structurally satisfied. Then we present a design method of the polynomial half-band kernel with the specified degree of flatness, in which the PBP is utilized to ensure that the polynomial half-band kernel has the specified zeros at $\lambda = 2$. Furthermore, we apply the Remez exchange algorithm to minimize the spectral error in stopband. It is well-known that the Remez exchange algorithm is an efficient approach for designing FIR linear phase filters with an equiripple magnitude response. The Remez exchange algorithm has been also used to design FIR linear phase half-band filters in [6]. In the proposed method using the Remez exchange algorithm, a set of coefficients is easily obtained only by solving a system of linear equations. The optimal solution is attained through a few iterations. Therefore, the proposed design algorithm is computationally efficient. Finally, two design examples are shown to demonstrate the effectiveness of the design method proposed in this paper.

2. PRELIMINARIES

We first give a brief review of signal processing on graphs in [7], [8] and [15]. A graph is denoted as $G = (\mathcal{V}, E)$, where $\mathcal{V}$ is the set of vertices (nodes) and E is the set of edges (links). The size of graph $N = |\mathcal{V}|$ is the number of vertices. $\mathbf{A}$ is the adjacency matrix, whose element $A(i, j)$ represents the weight of the edge between vertex i and j, and $A(i, j) = 0$ if there is no edge. $\mathbf{D} = \mathrm{diag}(d_i)$ is the diagonal degree matrix, where $d_i = \sum_j A(i, j)$ is the sum of weights of all edges connected to vertex i. The Laplacian matrix of the graph is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, and the normalized Laplacian matrix is $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$, where $\mathbf{I}$ is the identity matrix. Both $\mathbf{L}$ and $\mathcal{L}$ are symmetric positive semidefinite matrices, and have a complete set of orthonormal eigenvectors. Now

This work was supported by JSPS KAKENHI Grant Number 26330128.

we denote the eigenvectors of the normalized Laplacian matrix $\mathcal{L}$ by $\mathbf{u}_i = [u_i(1), u_i(2), \cdots, u_i(N)]^T$ and the associated eigenvalues by λ_i, where $u_i(n)$ is real-valued and $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N \le 2$.

A graph signal is a function defined on the graph and the sample value $f(n)$ at vertex n can be represented as a vector $\mathbf{f} = [f(1), f(2), \cdots, f(N)]^T$. The graph Fourier transform (GFT) is defined as the projections of a signal $\mathbf{f}$ on the graph onto the eigenvectors;

$$F(\lambda_i) = \mathbf{f}^T \mathbf{u_i} = \sum_{n=1}^{N} f(n) u_i(n) \tag{1}$$

and the inverse graph Fourier transform (IGFT) is given by

$$f(n) = \mathbf{F}^T \mathbf{U}(n) = \sum_{i=1}^{N} F(\lambda_i) u_i(n) \tag{2}$$

where $\mathbf{F} = [F(\lambda_1), F(\lambda_2), \cdots, F(\lambda_N)]^T$ and $\mathbf{U}(n) = [u_1(n), u_2(n), \cdots, u_N(n)]^T$.

A filtering operation of a signal $\mathbf{f}$ in the vertex domain can be expressed in the matrix form as $\mathbf{y} = \mathbf{H}\mathbf{f}$, where $\mathbf{y} = [y(1), y(2), \cdots, y(N)]^T$ is output signal and $\mathbf{H}$ is the transform matrix of the filter given by

$$\mathbf{H} = \sum_{i=1}^{N} H(\lambda_i) \mathbf{u}_i \mathbf{u}_i^T \tag{3}$$

where $H(\lambda)$ is the spectral kernel of the filter.

By using GFT, we have in the spectral domain

$$Y(\lambda_i) = H(\lambda_i) F(\lambda_i) \tag{4}$$

where $Y(\lambda_i)$ is the GFT of output signal $\mathbf{y}$.

3. GRAPH WAVELET FILTER BANKS

The two channel graph wavelet filter bank $\{\mathbf{H}_k, \mathbf{G}_k\}_{k=0,1}$ proposed in [15] is shown in Fig.1. The corresponding transform matrices are given by

$$\begin{cases} \mathbf{H}_k = \sum_{i=1}^{N} H_k(\lambda_i) \mathbf{u}_i \mathbf{u}_i^T \\ \mathbf{G}_k = \sum_{i=1}^{N} G_k(\lambda_i) \mathbf{u}_i \mathbf{u}_i^T \end{cases} \tag{5}$$

where $H_k(\lambda), G_k(\lambda)$ are the spectral kernels of analysis and synthesis filters, respectively. $\mathbf{H}_0, \mathbf{G}_0$ act as lowpass filters and $\mathbf{H}_1, \mathbf{G}_1$ are highpass. The down-sampling operation β_L discards the output coefficients of lowpass channel in the set H, while β_H discards the output coefficients of highpass channel in the set L, where $|H| + |L| = N$ and $H \cap L = 0$. The overall transform matrix of the filter bank is given by

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\{\mathbf{G}_0(\mathbf{I} + \mathbf{J}_\beta)\mathbf{H}_0 + \mathbf{G}_1(\mathbf{I} - \mathbf{J}_\beta)\mathbf{H}_1\} \\ &= \frac{1}{2}\{(\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1) + (\mathbf{G}_0\mathbf{J}_\beta\mathbf{H}_0 - \mathbf{G}_1\mathbf{J}_\beta\mathbf{H}_1)\} \end{aligned} \tag{6}$$

where $\mathbf{J}_\beta = \mathrm{diag}(\beta_i)$ is a diagonal matrix, and β_i is a partition function such that $\beta_i = 1$ if vertex $i \in L$ and $\beta_i = -1$ if vertex $i \in H$. Thus the down-and-up sampling operation β_L in lowpass channel can be expressed in the matrix form as $\frac{1}{2}(\mathbf{I} + \mathbf{J}_\beta)$, while β_H in highpass channel as $\frac{1}{2}(\mathbf{I} - \mathbf{J}_\beta)$. It is shown in [15] and [17] that the perfect reconstruction (PR) condition of two channel filter bank is given by

$$\begin{cases} H_0(\lambda)G_0(\lambda) + H_1(\lambda)G_1(\lambda) = 2 \\ H_0(2-\lambda)G_0(\lambda) - H_1(2-\lambda)G_1(\lambda) = 0 \end{cases} \tag{7}$$

To cancel aliasing, synthesis kernels are chosen as

$$\begin{cases} G_0(\lambda) = H_1(2-\lambda) \\ G_1(\lambda) = H_0(2-\lambda) \end{cases} \tag{8}$$

Therefore, the PR condition in Eq.(7) can be reduced to

$$H_0(\lambda)H_1(2-\lambda) + H_0(2-\lambda)H_1(\lambda) = 2 \tag{9}$$

which leads to a biorthogonal filter bank.

Furthermore, defining the product filter $P(\lambda)$ as

$$P(\lambda) = H_0(\lambda)H_1(2-\lambda) \tag{10}$$

thus, Eq.(9) becomes

$$P(\lambda) + P(2-\lambda) = 2 \tag{11}$$

It is known in [17] that $P(\lambda)$ should be a half-band kernel and an odd degree polynomial to satisfy Eq.(11). It is also a lowpass kernel since $H_0(\lambda)$ and $G_0(\lambda) = H_1(2-\lambda)$ are lowpass.

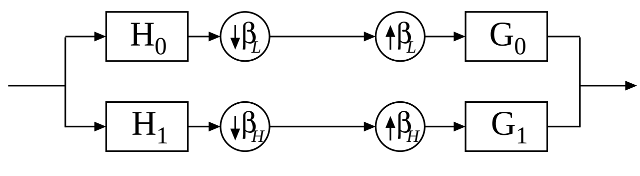

Fig. 1. Two channel graph wavelet filter bank.

4. DESIGN OF BIORTHOGONAL GRAPH WAVELET FILTER BANKS

It is known in [15] and [17] that if the spectral kernel is a polynomial of degree K, then the graph filter is exactly K-hop localized and can be implemented iteratively with K one-hop operations at each vertex without any matrix diagonalization. In this paper, we will discuss the polynomial approximation of the desired kernel.

4.1. Construction of Filter Bank

Firstly, we consider the design of $H_0(\lambda)$. $H_0(\lambda)$ is a lowpass kernel and has a desired gain $\sqrt{2}$ in passband $[0, \lambda_p]$ and 0 in stopband $[\lambda_s, 2]$. We construct $H_0(\lambda)$ as

$$H_0(\lambda) = \frac{1}{\sqrt{2}}(1 + Q_0(\lambda)) \tag{12}$$

and

$$Q_0(\lambda) = \sum_{k=0}^{K_0} a_0(k)(\lambda - 1)^{2k+1} \tag{13}$$

where the degree of polynomial $Q_0(\lambda)$ is $2K_0 + 1$ and $a_0(k)$ are real-valued coefficients.

Since $Q_0(\lambda) = -Q_0(2-\lambda)$, we have

$$H_0(\lambda) + H_0(2-\lambda) = \sqrt{2} \tag{14}$$

which means that $H_0(\lambda)$ defined in Eq.(12) is a polynomial half-band kernel.

We then construct the highpass kernel $H_1(\lambda)$ as

$$H_1(\lambda) = \sqrt{2} - Q_1(\lambda)H_0(\lambda) \tag{15}$$

and

$$Q_1(\lambda) = \sum_{k=0}^{K_1} a_1(k)(\lambda-1)^{2k+1} \tag{16}$$

where the degree of $Q_1(\lambda)$ is $2K_1 + 1$ and $a_1(k)$ are real-valued.

Therefore, we have

$$P(\lambda) = H_0(\lambda)H_1(2-\lambda)$$

$$= 1 + Q_0(\lambda) + \frac{1}{2}Q_1(\lambda) - \frac{1}{2}Q_0^2(\lambda)Q_1(\lambda) \tag{17}$$

It is clear that the PR condition in Eq.(11) is satisfied regardless of what the coefficients of $Q_0(\lambda)$ and $Q_1(\lambda)$ are. That is, the PR condition is structurally satisfied.

For $H_0(\lambda)$ in Eq.(12) to be lowpass, we must have

$$Q_0(\lambda) = \begin{cases} 1 & (0 \leq \lambda \leq \lambda_p) \\ -1 & (\lambda_s \leq \lambda \leq 2) \end{cases} \tag{18}$$

where $\lambda_p + \lambda_s = 2$. Due to the antisymmetry of $Q_0(\lambda)$, its desired spectral is

$$Q_0^d(\lambda) = 1 \qquad (0 \leq \lambda \leq \lambda_p). \tag{19}$$

Further, $H_1(\lambda)$ in Eq.(15) should be highpass, that is, $H_1(\lambda) = 0$ in $[0, \lambda_p]$ and $H_1(\lambda) = \sqrt{2}$ in $[\lambda_s, 2]$. In $[\lambda_s, 2]$, $H_0(\lambda) = 0$, thus we have $H_1(\lambda) = \sqrt{2}$. On the other hand, since $H_0(\lambda) = \sqrt{2}$ in $[0, \lambda_p]$ ideally, then the desired spectral of $Q_1(\lambda)$ is the same as $Q_0(\lambda)$;

$$Q_1^d(\lambda) = 1 \qquad (0 \leq \lambda \leq \lambda_p). \tag{20}$$

However, $H_0(\lambda)$ has some errors in practice. Thus we have from Eq.(15)

$$Q_1^d(\lambda) = \frac{\sqrt{2}}{H_0(\lambda)} \qquad (0 \leq \lambda \leq \lambda_p). \tag{21}$$

That is, the actual response of $H_0(\lambda)$ should be considered in the design of $H_1(\lambda)$. In the following, we will discuss the design of $Q_0(\lambda)$ and $Q_1(\lambda)$ (that is, $H_0(\lambda)$ and $H_1(\lambda)$).

4.2. Approximation of Half-band Kernel

The PBP (Parametric Bernstein Polynomial) was first introduced in [4], and expressed in [20] and [6] as

$$B(x) = K(x) - \sum_{i=L}^{(N_b-1)/2} \alpha_i K_i(x) \tag{22}$$

and

$$K(x) = \sum_{i=0}^{(N_b-1)/2} \binom{N_b}{i} x^i (1-x)^{N_b-i} \tag{23}$$

$$K_i(x) = \binom{N_b}{i} \{x^i(1-x)^{N_b-i} - x^{N_b-i}(1-x)^i\} \tag{24}$$

where the coefficients α_i are real, the degree N_b is odd, and L is integer and $0 \leq L < (N_b+1)/2$. It is clear that $B(x)$ is a halfband polynomial since $B(x) + B(1-x) = 1$, and $B(x)$ has L zeros at $x = 1$. If $\alpha_i = 0$ for all i, then the maximally flat response is obtained, that is, $L_{max} = (N_b+1)/2$.

By using the PBP $B(x)$, we define

$$H_0(\lambda) = \sqrt{2}B(\frac{\lambda}{2}) = \sqrt{2}\{K(\frac{\lambda}{2}) - \sum_{i=L_0}^{K_0} \alpha_i K_i(\frac{\lambda}{2})\} \tag{25}$$

where $N_0 = 2K_0 + 1$ and thus $H_0(\lambda)$ has L_0 zeros at $\lambda = 2$. Therefore, we have from Eq.(12)

$$Q_0(\lambda) = 2B(\frac{\lambda}{2}) - 1 = 2\{K(\frac{\lambda}{2}) - \sum_{i=L_0}^{K_0} \alpha_i K_i(\frac{\lambda}{2})\} - 1 \tag{26}$$

Next, we use the method in [6] to design $H_0(\lambda)$. To obtain a sharper spectral , we use the remaining degree of freedom for $B(\frac{\lambda}{2})$ to satisfy

$$1 - \delta \leq B(\frac{\lambda}{2}) \leq 1 + \delta \qquad (0 \leq \lambda \leq \lambda_p) \tag{27}$$

and

$$-\delta \leq B(\frac{\lambda}{2}) \leq \delta \qquad (\lambda_s \leq \lambda \leq 2) \tag{28}$$

where δ is a tolerance error. From Eq.(26), Eqs.(27) and (28) can be reduced to

$$1 - 2\delta \leq Q_0(\lambda) \leq 1 + 2\delta \qquad (0 \leq \lambda \leq \lambda_p). \tag{29}$$

By applying the Remez exchange algorithm, we select (M_0+1) extremal points λ_m as $\lambda_p = \lambda_0 > \lambda_1 > \cdots > \lambda_{M_0} > 0$, where $M_0 = K_0 - L_0 + 1$, and then formulate $Q_0(\lambda)$ as

$$Q_0(\lambda_m) = 1 - (-1)^m 2\delta. \tag{30}$$

By substituting $Q_0(\lambda)$ in Eq.(26) into Eq.(30), we derive a system of linear equations as follows;

$$\sum_{i=L_0}^{K_0} \alpha_i K_i(\frac{\lambda_m}{2}) - (-1)^m \delta = K(\frac{\lambda_m}{2}) - 1 \tag{31}$$

for $m = 0, 1, \cdots, M_0$. It is clear that there are (M_0+1) equations with respect to $M_0 = K_0 - L_0 + 1$ unknown coefficients α_i plus one error δ. Therefore, we can solve Eq.(31) to obtain a set of coefficients α_i. Since the extremal points λ_m are unknown a priori, we initially select λ_m equally spaced in $[0, \lambda_p]$, and then utilize an iteration procedure to obtain the equiripple spectral. Since it only needs to solve a system of linear equations iteratively, the proposed design algorithm is computationally efficient.

On the other hand, since the desired spectral of $Q_1(\lambda)$ is dependent on $H_0(\lambda)$ as shown in Eq.(21), it is needed to satisfy

$$\sqrt{2} - \delta \leq H_0(\lambda)Q_1(\lambda) \leq \sqrt{2} + \delta \qquad (0 \leq \lambda \leq \lambda_p). \tag{32}$$

Similarly, we apply the Remez exchange algorithm in $[0, \lambda_p]$ and formulate $Q_1(\lambda)$ as

$$H_0(\lambda_m)Q_1(\lambda_m) = \sqrt{2} - (-1)^m \delta. \tag{33}$$

Therefore, we can obtain

$$\sum_{i=L_1}^{K_1} \alpha_i K_i(\frac{\lambda_m}{2}) - \frac{(-1)^m}{2H_0(\lambda_m)}\delta = K(\frac{\lambda_m}{2}) - \frac{1}{2} - \frac{1}{\sqrt{2}H_0(\lambda_m)} \tag{34}$$

where the stopband error of $H_1(\lambda)$ is minimized regarding the actual response of $H_0(\lambda)$.

5. DESIGN EXAMPLES

In this section, we present two design examples to demonstrate the effectiveness of the proposed design method.

Example 1: We have designed the proposed biorthogonal graph wavelet filter banks with the maximally flat half-band spectral kernels. We have chosen $\{K_0, K_1\} = \{8, 6\}, \{12, 6\}, \{16, 6\}$. The degree of $H_0(\lambda)$ and $H_1(\lambda)$ are $2K_0 + 1$ and $2(K_0 + K_1) + 2$ respectively. The obtained spectral responses of $H_0(\lambda)$ and $H_1(\lambda)$ are shown in Fig. 2. For comparison, the spectral responses of biorthogonal graph wavelet filter bank *graphBior*$(8, 8)$ proposed in [17] by Narang and Ortega have also been shown in Fig. 2, where the degree of $H_0(\lambda)$ and $H_1(\lambda)$ are 16 and 15 respectively. It is seen that the spectral responses of the proposed filter banks are flatter at $\lambda = 0$ and $\lambda = 2$ than the *graphBior*, and become sharper nearby $\lambda = 1$ with an increasing K_0.

Example 2: We have designed the proposed biorthogonal graph wavelet filter bank with $K_0 = K_1 = 10$, and $\lambda_p = 0.8$, $\lambda_s = 1.2$. We have chosen $L_0 = 11, 9, 7$ and designed $H_0(\lambda)$. The spectral responses of $H_0(\lambda)$ with $L_0 = 11, 9, 7$ are shown in Fig. 3. Note that $L_0 = 11$ means the maximally flat half-band spectral kernel. It is seen that the spectral error of $H_0(\lambda)$ become smaller with an decreasing L_0. We then designed $H_1(\lambda)$ with $L_1 = 5$, and the spectral responses of $H_1(\lambda)$ are shown in Fig. 3 also. It is seen that the equiripple spectral responses of $H_1(\lambda)$ have been obtained in the stopband by using the Remez exchange algorithm. However, $H_1(\lambda)$ has an overshooting above $\lambda = 1$.

6. CONCLUSION

In this paper, we have proposed a new class of critically sampled compactly supported biorthogonal graph wavelet filter banks with flat spectral responses. We have used the polynomial half-band kernels to construct biorthogonal graph filter banks, which structurally safisfy the perfect reconstruction condition. We then have presented a design method for the polynomial half-band kernel, in which the PBP is utilized to ensure that the polynomial half-band kernel has the specified zeros at $\lambda = 2$. Furthermore, we have applied the Remez exchange algorithm to minimize the spectral error in stopband. In the proposed method, a set of coefficients can be easily obtained only by solving a system of linear equations. The optimal solution is attained through a few iterations. Therefore, the proposed design algorithm is computationally efficient. Finally, two design examples have been shown to demonstrate the effectiveness of the design method proposed in this paper.

7. REFERENCES

[1] I. Daubechies, *Ten Lectures on Wavelets*, Philadelphia, PA: SIAM, 1992.

[2] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs, NJ: Prentice Hall, 1993.

[3] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Cambridge, MA, USA: Wellesley-Cambridge Press, 1996.

[4] H. Caglar and A. N. Akansu, "A generalized parametric PR-QMF design technique based on Bernstein polynomial approximation," *IEEE Trans. Signal Process.*, vol.41, no.7, pp.2314–2321, July 1993.

[5] D. B. H. Tay, "Zero-pinning the Bernstein polynomial: A simple design technique for orthonormal wavelets," *IEEE Signal Process. Lett.*, vol.12, no.12, pp.835–838, Dec. 2005.

[6] X. Zhang, "Design of FIR halfband filters for orthonormal wavelets using Remez exchange algorithm," *IEEE Signal Process. Lett.*, vol.16, no.9, pp.814–817, Sep. 2009.

[7] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega and P. Vandergheynst, "The emerging field of signal processing on graphs," *IEEE Signal Process. Mag.*, pp.83–98, May 2013.

[8] A. Sandryhaila and J. M. F. Moura, "Big data analysis with signal processing on graphs," *IEEE Signal Process. Mag.*, pp.80–90, Sep. 2014.

[9] M. Crovella and E. Kolaczyk, "Graph wavelets for spatial traffic analysis," in Proc. INFOCOM 2003, vol.3, pp.1848–1857, Mar. 2003.

[10] W. Wang and K. Ramchandran, "Random multiresolution representations for arbitrary network graphs," in Proc. ICASSP 2006, vol.4, pp.161–164, May 2006.

[11] R. Coifman and M. Maggioni, "Diffusion wavelets," Appl. Comput. Harmon. Anal., vol.21, pp.53–94, 2006.

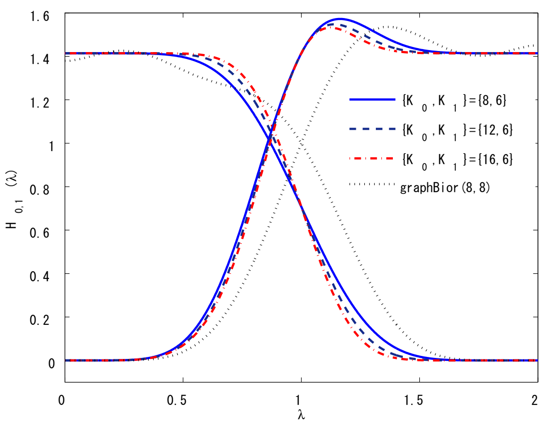

Fig. 2. Spectral responses of $H_0(\lambda)$ and $H_1(\lambda)$ in Example 1.

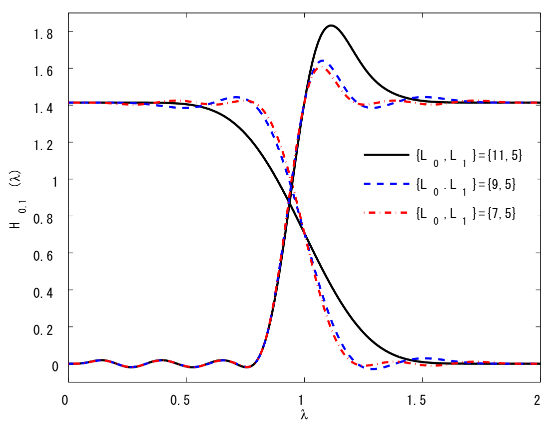

Fig. 3. Spectral responses of $H_0(\lambda)$ and $H_1(\lambda)$ in Example 2.

[12] G. Shen and A. Ortega, "Optimized distributed 2D transforms for irregularly sampled sensor network grids using wavelet lifting," in Proc. ICASSP 2008, pp. 2513–2516, Apr. 2008.

[13] S. K. Narang and A. Ortega, "Lifting based wavelet transforms on graphs," in Proc. APSIPAASC 2009, pp.441–444, Oct. 2009.

[14] D. K. Hammond, P. Vandergheynst and R. Gribonval, "Wavelets on graphs via spectral graph theory," Appl. Comput. Harmon. Anal., vol.30, no.2, pp.129–150, 2011.

[15] S. K. Narang and A. Ortega, "Perfect reconstruction two-channel wavelet filter banks for graph structured data," *IEEE Trans. Signal Process.*, vol.60, no.6, pp.2786–2799, June 2012.

[16] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs," *IEEE Trans. Signal Process.*, vol.61, no.7, pp.1644–1636, Apr. 2013.

[17] S. K. Narang and A. Ortega, "Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs," *IEEE Trans. Signal Process.*, vol.61, no.19, pp.4673–4685, Oct. 2013.

[18] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs: Frequency analysis," *IEEE Trans. Signal Process.*, vol.62, no.12, pp.3042–3054, June 2014.

[19] D. B. H. Tay and Z. P. Lin, "Design of near orthogonal graph filter banks," *IEEE Signal Processing Letters*, vol.22, no.6, pp.701–704, June 2015.

[20] D. B. H. Tay and Z. P. Lin, "Graph QMF with flatness constraints," in Proc. ISCAS 2015, pp. 2600–2603, May 2015.