CLOSED-FORM DESIGN OF MAXIMALLY FLAT IIR HALF-BAND FILTERS

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ABSTRACT

In this paper, a new closed-form expression for the transfer function of the maximally flat (MF) IIR half-band (HB) filters is presented. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the maximal flatness conditions. The proposed IIR half-band filters are more general than the existing half-band filters, because they include the conventional FIR half-band filters with exactly linear phase, the generalized FIR half-band filters with approximately linear phase, and the allpass-based IIR half-band filters, as special cases. Furthermore, the IIR HB filters with exactly linear phase and the causal stable IIR HB filters can be realized also. Finally, some design examples are presented to demonstrate the effectiveness of the proposed IIR HB filters.

1. INTRODUCTION

Half-band (HB) filters are of great importance and are often used in multirate digital signal processing systems, filter banks and wavelets [1]∼[5],[9]. In many applications such as the wavelet-based image coding, HB filters are required to possess the maximally flat (MF) frequency response in order to get better coding performance. Much work has been done, which is mainly devoted to the design of FIR HB filters. The closed-form solution for the MF FIR HB filters with exactly linear phase can be found in [1]∼[3], while that for the generalized MF FIR HB filters with approximately linear phase is recently presented in [8]∼[10]. In contrast, there exists little work regarding IIR HB filters. A class of IIR HB filters is given in [2] and [5] by using the parallel structure of a pure delay section and an allpass subfilter. It has been shown also in [5] that the design of such allpass-based IIR HB filters can be reduced to designing the corresponding all-pole filters. The closed-form solution for the allpass-based MF IIR HB filters is given in [3],[5].

In this paper, we propose a class of more general IIR HB filters than the existing HB filters. The proposed IIR HB filters include not only the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase, and the allpass-based IIR HB filters as special cases, but also the causal stable IIR HB filters and the IIR HB filters with exactly linear phase. We then present a new closed-form expression for the transfer function of the MF IIR HB filters. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the maximal flatness conditions. We also investigate the conditions for realizing the causal stable IIR HB filters. Finally, some design examples are presented to demonstrate the effectiveness of the proposed method.

2. IIR HB FILTERS

Let h_n $(n = 0, 1, \cdots)$ be the impulse response of an IIR HB filter. It is well-known that the impulse response h_n should satisfy

$$\begin{cases} h_K = \dfrac{1}{2} & \\ h_{K+2k} = 0 & (k = \pm 1, \pm 2, \cdots) \end{cases}, \tag{1}$$

where K is odd. Then the transfer function $H(z)$ of the IIR HB filter can be given by

$$H(z) = \frac{1}{2} z^{-K} + G(z^2). \tag{2}$$

If $G(z)$ is a FIR filter, then $H(z)$ becomes a FIR HB filter. In this paper, we assume that $G(z)$ is a general IIR filter with numerator degree N and denominator degree M;

$$G(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}}, \tag{3}$$

where a_n and b_m are real coefficients, and $b_0 = 1$. It is noted that if $M = 0$, $G(z)$ will degenerate into a FIR filter. Then $H(z)$ becomes the generalized FIR HB filters proposed in [8]∼[10]. If we further impose the constraints of $N = K$ and $a_n = a_{N-n}$ on $G(z)$, then $H(z)$ will be the conventional FIR HB filters with exactly linear phase. In addition, if $N = M$ and $b_n = 2a_{N-n}$, then $G(z)$ will be allpass with gain $1/2$, and $H(z)$ are the allpass-based IIR HB filters presented in [5], which are based on the parallel structure of a pure delay section z^{-K} and an allpass subfilter. To summarize, the IIR HB filters proposed in this paper include the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters, and the allpass-based IIR HB filters as special cases, and then are more general than the existing HB filters.

3. FILTER PROPERTIES

Let $\hat{H}(z)$ be the advanced version of $H(z)$, that is,

$$\hat{H}(z) = z^K H(z) = \frac{1}{2} + \frac{\sum_{n=0}^{N} a_n z^{K-2n}}{\sum_{m=0}^{M} b_m z^{-2m}}, \tag{4}$$

then there exist the following relations between $H(z)$ and $\hat{H}(z)$;

$$\begin{cases} |H(e^{j\omega})| = |\hat{H}(e^{j\omega})| \\ \theta(\omega) = -K\omega + \hat{\theta}(\omega) \\ \tau(\omega) = K + \hat{\tau}(\omega) \end{cases}, \tag{5}$$

where $\theta(\omega)$, $\hat{\theta}(\omega)$ and $\tau(\omega)$, $\hat{\tau}(\omega)$ are the phase and group delay responses of $H(z)$ and $\hat{H}(z)$, respectively. The frequency response of $\hat{H}(z)$ is given from Eq.(4) by

$$\hat{H}(e^{j\omega}) = \frac{1}{2} + \frac{\sum_{n=0}^{N} a_n e^{j(K-2n)\omega}}{\sum_{m=0}^{M} b_m e^{-j2m\omega}}, \tag{6}$$

which satisfies

$$\hat{H}(e^{j\omega}) + \hat{H}^*(e^{j(\pi-\omega)}) \equiv 1, \tag{7}$$

where x^* denotes the complex conjugate of x.

For the maximally flat (MF) design, $H(z)$ is required to satisfy the following flatness condition;

$$\left.\frac{\partial^i |H(e^{j\omega})|}{\partial\omega^i}\right|_{\omega=\pi} = 0 \qquad (i = 0, 1, \cdots, N+M). \tag{8}$$

This means that $H(z)$ must have $N+M+1$ zeros located at $z=-1$. Then we have

$$\left.\frac{\partial^i \hat{H}(e^{j\omega})}{\partial\omega^i}\right|_{\omega=\pi} = 0 \qquad (i = 0, 1, \cdots, N+M). \tag{9}$$

It can be derived from Eq.(7) that $\hat{H}(z)$ satisfies also

$$\begin{cases} \hat{H}(1) = 1 \\ \left.\frac{\partial^i \hat{H}(e^{j\omega})}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \cdots, N+M) \end{cases}, \tag{10}$$

that is,

$$\begin{cases} |\hat{H}(1)| = 1 \\ \left.\frac{\partial^i |\hat{H}(e^{j\omega})|}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \cdots, N+M) \end{cases}, \tag{11}$$

and

$$\begin{cases} \left.\frac{\partial^i \hat{\theta}(\omega)}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \cdots, N+M) \\ \left.\frac{\partial^i \hat{\tau}(\omega)}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \cdots, N+M-1) \end{cases}. \tag{12}$$

Therefore, we can get from Eq.(5)

$$\begin{cases} |H(1)| = 1 \\ \left.\frac{\partial^i |H(e^{j\omega})|}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \cdots, N+M) \end{cases}, \tag{13}$$

and

$$\begin{cases} \tau(0) = K \\ \left.\frac{\partial^i \tau(\omega)}{\partial\omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \cdots, N+M-1) \end{cases}. \tag{14}$$

That is, both the magnitude and group delay responses of $H(z)$ are maximally flat also at $\omega = 0$.

4. CLOSED-FORM SOLUTION

In this section, we give a new closed-form solution for the MF IIR HB filters. We have from Eq.(6)

$$\hat{H}(e^{j\omega}) = \frac{N(\omega)}{D(\omega)}, \tag{15}$$

where

$$\begin{cases} N(\omega) = \sum_{n=0}^{N} a_n e^{j(K-2n)\omega} + \frac{1}{2}\sum_{m=0}^{M} b_m e^{-j2m\omega} \\ D(\omega) = \sum_{m=0}^{M} b_m e^{-j2m\omega} \end{cases}. \tag{16}$$

Since $H(z)$ has $N+M+1$ zeros located at $z=-1$, the flatness condition in Eq.(9) is equivalent to

$$\left.\frac{\partial^i N(\omega)}{\partial\omega^i}\right|_{\omega=\pi} = 0 \qquad (i = 0, 1, \cdots, N+M). \tag{17}$$

From Eq.(16), we have

$$\left.\frac{\partial^i N(\omega)}{\partial\omega^i}\right|_{\omega=\pi} = \frac{1}{2}\sum_{m=0}^{M} b_m(-j2m)^i - \sum_{n=0}^{N} a_n(j(K-2n))^i. \tag{18}$$

Substituting Eq.(18) into Eq.(17), we get

$$2\sum_{n=0}^{N} a_n(K-2n)^i - \sum_{m=0}^{M} b_m(-2m)^i = 0. \tag{19}$$

where $i = 0, 1, \cdots, N+M$. Since $b_0 = 1$, we rewrite Eq.(19) in matrix form as

$$\boldsymbol{VDa} = \boldsymbol{u}, \tag{20}$$

where $\boldsymbol{a} = [a_0, a_1, \cdots, a_N, b_1, \cdots, b_M]^T$, $\boldsymbol{u} = [1, 0, \cdots, 0]^T$,

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ K & K-2 & \cdots & K-2N & -2 & \cdots & -2M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K^{N+M} & (K-2)^{N+M} & \cdots & (K-2N)^{N+M} & (-2)^{N+M} & \cdots & (-2M)^{N+M} \end{bmatrix}, \tag{21}$$

and $\boldsymbol{D} = \mathrm{diag}[d_0, d_1, \cdots, d_{N+M}]$,

$$d_i = \begin{cases} 2 & (0 \le i \le N) \\ -1 & (N+1 \le i \le N+M) \end{cases} . \tag{22}$$

It should be noted that $\boldsymbol{V}$ is the Vandermonde matrix. Therefore, there is always a unique solution and a closed-form solution can be obtained by

$$\begin{cases} a_n = \dfrac{(-1)^{N-n}}{2} \dbinom{N}{n} \dfrac{M!}{N!} \dfrac{\prod_{i=0}^{N}(\frac{K}{2} - i)}{\prod_{i=0}^{M}(\frac{K}{2} + i - n)} \\ b_m = \dbinom{M}{m} \prod_{i=1}^{m} \dfrac{N+1-\frac{K}{2}-i}{\frac{K}{2}+i} \end{cases} . \tag{23}$$

5. RELATION WITH THE EXISTING HB FILTERS

In this section, we examine the relationship between the proposed MF IIR HB filters and the existing MF HB filters, and present some new MF IIR HB filters, such as the exactly linear phase IIR HB filters and the causal stable IIR HB filters.

5.1. The generalized FIR HB filters

When $M = 0$, $G(z)$ is a FIR filter, and then $H(z)$ becomes FIR HB filter. In this case, the closed-form solution for the MF HB filters can be obtained by substituting $M = 0$ into Eq.(23), that is,

$$a_n = \frac{(-1)^n}{(2n-K)} \frac{\prod_{i=0}^{N}(i - \frac{K}{2})}{n!(N-n)!}, \tag{24}$$

which is the same as that for the generalized MF FIR HB filters presented in [8]∼[10]. If $N = K$ (where K is odd),

$$a_n = \frac{(-1)^{(N+1)/2+n}}{(2n-N)} \frac{\prod_{i=0}^{(N-1)/2}(i + \frac{1}{2})^2}{n!(N-n)!}. \tag{25}$$

It is easily verified that the condition of $a_n = a_{N-n}$ is satisfied. That is, the conventional MF FIR HB filters with exactly linear phase are obtained.

5.2. The allpass-based IIR HB filters

If we assume that $N = M$, then Eq.(23) becomes

$$\begin{cases} a_n = \dfrac{1}{2} \dbinom{N}{n} \prod_{i=1}^{N-n} \dfrac{N+1-\frac{K}{2}-i}{\frac{K}{2}+i} \\ b_n = \dbinom{N}{n} \prod_{i=1}^{n} \dfrac{N+1-\frac{K}{2}-i}{\frac{K}{2}+i} \end{cases} , \tag{26}$$

which satisfy the condition of $b_n = 2a_{N-n}$. Therefore, $G(z)$ is an allpass filter, and $H(z)$ is the allpass-based IIR HB filter that is the same as in [5]. As shown in [5] and [6], the allpass-based IIR HB filters may be unstable depending on K. To get a causal stable HB filter, K should be chosen to satisfy

$$K > 2(N-1). \tag{27}$$

This is because a causal stable allpass filter of order N has a monotonically decreasing phase response, and its phase is $-N\pi$ at $\omega = \pi$ [6].

5.3. The exactly linear phase IIR HB filters

Assume that N is odd and M is even. Let $K = N - M$, then

$$\begin{cases} a_n = \dfrac{(-1)^n}{2} \dfrac{M!}{n!(N-n)!} \dfrac{\prod_{i=0}^{N}(\frac{N}{2} - i - \frac{M}{2})}{\prod_{i=0}^{M}(i - \frac{M}{2} + n - \frac{N}{2})} \\ b_m = (-1)^m \dbinom{M}{m} \prod_{i=0}^{N} \dfrac{\frac{N}{2}-i-\frac{M}{2}}{\frac{N}{2}-i+m-\frac{M}{2}} \end{cases} , \tag{28}$$

which satisfy the conditions of $a_n = a_{N-n}$ and $b_m = b_{M-m}$. That is, the exactly linear phase IIR HB filters are realized. It is noted that such IIR HB filters are unstable, and have to be divided into the causal stable and anticausal stable parts to implement. When $M = 0$, they will degenerate into the conventional FIR HB filters with exactly linear phase.

5.4. The causal stable IIR HB filters

The proposed IIR HB filters may be unstable depending on K. To guarantee the filters to be causal stable, we have to choose a larger K. Like the allpass-based IIR HB filters, there exists K_{min} so that when $K \ge K_{min}$, the MF IIR HB filters are causal stable. In general, K_{min} is dependent on N and M. For example, when $M = 1$, $G(z)$ has a pole of $z_p = \frac{K-2N}{K+2}$. To get a causal stable $G(z)$, the pole must be located inside the unit circle, that is,

$$|z_p| = |\frac{K-2N}{K+2}| < 1. \tag{29}$$

To satisfy Eq.(29), we should have

$$K > N - 1. \tag{30}$$

Then, $K_{min} = N$ if N is odd, and $K_{min} = N + 1$ if N is even. For $M \ge 2$, it is more complicated to determine K_{min}, and further investigation is needed.

6. DESIGN EXAMPLES

Many design examples of the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase, and the allpass-based IIR HB filters have been given in [1]∼[5], [8]∼[10]. In this paper, we present some examples of general MF IIR HB filters only. We consider the IIR HB filters with $N = 5$ and $M = 2$. Firstly, we set $K = 9$ and obtain a set of filter coefficients from Eq.(23). The impulse response of the obtained MF IIR HB filter is shown in Fig.1. It is clear that this filter is causal stable. The resulting magnitude and group delay responses are shown by the solid line in Fig.2 and in Fig.3, respctively. We have also designed other IIR HB filters with various K, and found that $K_{min} = 7$. In addition, the IIR HB filter with $K = 3$ has an exactly linear phase, whose magnitude and group delay responses are shown in the dotted line in Fig.2 and Fig.3 also.

7. CONCLUSIONS

In this paper, we have proposed a more general class of IIR HB filters than the existing HB filters. The proposed IIR HB filters include not only the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase, and the allpass-based IIR HB filters as special cases, but also the causal stable IIR HB filters and the IIR HB filters with exactly linear phase. We have given a new closed-form expression for the transfer function of the MF IIR HB filters. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations from the maximal flatness conditions. We have also investigated the conditions for realizing the causal stable IIR HB filters and the IIR HB filters with exactly linear phase.

8. REFERENCES

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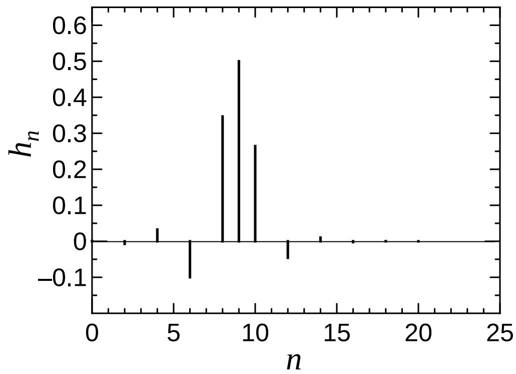

Fig. 1. Impulse response of $H(z)$ with $K = 9$

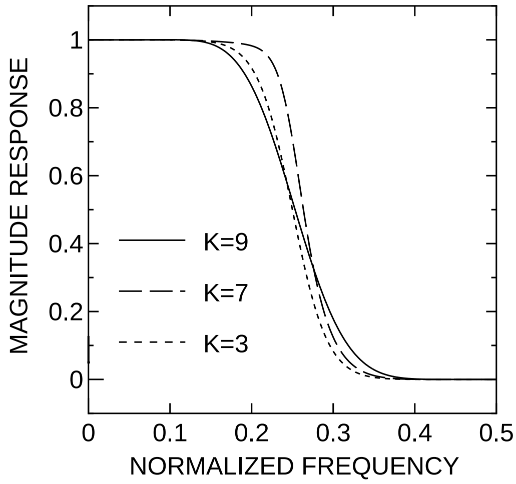

Fig. 2. Magnitude responses of $H(z)$

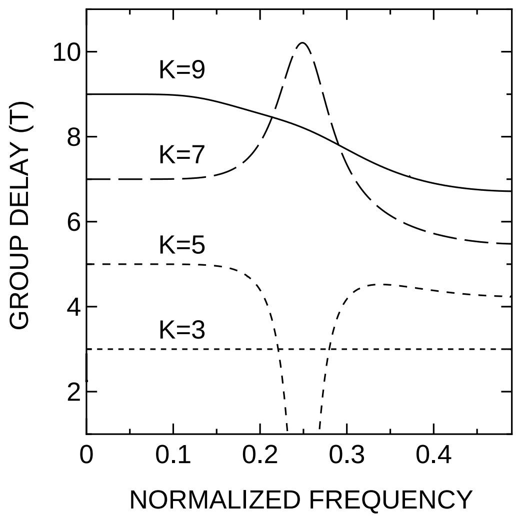

Fig. 3. Group delay responses of $H(z)$