Design of Orthogonal Graph Wavelet Filter Banks

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***Abstract*—This paper proposes a novel design method of two channel compactly supported near orthogonal graph wavelet filter banks. We firstly prove the necessary condition for the orthogonality of filter banks where the kernel filter must have a flat spectral in passband if having flat stopband, and then derive the closed-form solution for the maximally flat filters. We also give a filter function that ensures the specified degrees of flatness at $\lambda = 0$ and $\lambda = 2$ by the maximally flat filters. Therefore, we can use the remaining coefficients to minimize the reconstruction error of filter banks, where the optimization tool in Matlab is used to design the filter banks without any initial solution. Some examples are designed by using the proposed method and the spectral are compared with the conventional design methods. It is shown from the experimental results that the graph filter banks proposed in this paper outperform the conventional kernel filters. Finally, the proposed graph wavelets are applied to the Minnesota traffic graph to demonstrate the effectiveness of the proposed design method.**

I. Introduction

Signal processing on graphs has a wide range of applications such as social, sensor, biological, and transportation networks [3], [4]. Graph signal processing aims to extend the traditional signal processing theories and methodologies to signals defined on graphs. Major challenges are how to efficiently process and compress large amounts of signals on general graphs. It has been shown in [1], [2] that wavelet transforms provide a sparse representation of signals as a widely used signal processing tool. Recently, many works have been done to extend the traditional wavelet transform to the graph signals, namely, graph wavelet transform [5]∼[14]. However, the graph wavelet transforms proposed in [5]∼[7] and [10] are not critically sampled. It is known that critical sampling is of importance for compact representation of signals, e.g., compression. Critically sampled wavelet transforms have been also proposed in [8], [9], [11], [12], [13] and [14]. Furthermore, the graph wavelet filters are required to be compact-supported, that is, the output signal at each vertex is computed only from the signal at that vertex and its K-hop neighborhoods. It is achieved by a polynomial approximation of the spectral kernel of the filter. The wavelet filter banks based on the lifting scheme proposed in [8] and [9] and *graphBior* wavelet filter bank propose in [12] are compact-supported, but not orthogonal. The critically sampled orthogonal wavelet filter bank *graph-QMF* proposed in [11] cannot achieve the perfect reconstruction (PR) exactly by using the polynomial approximation. Narang and Ortega had proposed a simple design method by using the Meyer's wavelet construction to get near orthogonal graph QMFs in [11]. However, The reconstruction error of graph QMFs cannot be controlled directly and may be quite large. Tay and Lin had proposed a constrained optimization method to minimize the reconstruction error in [13], where the Bernstein polynomial was used to generate the initial solution for the nonlinear optimization. However, the resulting kernel filter has not a flat spectral in passband. In [14], the design of graph QMFs with flatness constraints has been discussed, where the flat spectral response in passband was considered also, but only one filter coefficient was used to reduce the reconstruction error.

This paper proposes a new method for designing two channel compact-supported near orthogonal graph wavelet filters with a flat spectral both in passband and stopband. Firstly, we prove the necessary condition for the orthogonality of graph wavelet filter banks in which the kernel filter must have a flat spectral also in passband if the stopband spectral is flat. We then derive the closed-form solution for the maximally flat filters having the specified degrees of flatness at $\lambda = 0$ and $\lambda = 2$. According to the resulting maximally flat filters, we give a kernel filter function that ensures the specified degrees of flatness at $\lambda = 0$ and $\lambda = 2$ by the maximally flat filters. Therefore, we can use the remaining coefficients to minimize the reconstruction error of wavelet filter banks. By using this kernel filter function, we can use the optimization tool in Matlab to design the graph QMFs without generating any initial solution, which is different from in [13]. The number of unknown coefficients in the kernel filter function is smaller than that in [13], thus the design method proposed in this paper is more computationally efficient. Some examples are designed by using the proposed method and compared with the method in [13]. It is shown from the experimental results that the proposed graph wavelet filters are better than the conventional ones. Finally, the proposed graph wavelet filters are applied to the Minnesota traffic graph in [10], [11] and the performance is compared with the conventional graph wavelets to demonstrate the effectiveness of the design method proposed in this paper.

II. Preliminaries

First of all, we briefly review the fundamental of signal processing on graphs in [3], [4] and [11]. A graph is denoted as $G = (\mathcal{V}, E)$, where $\mathcal{V}$ is the set of vertices or nodes and E is the set of edges or links. The size of graph is the number of vertices: $N = |\mathcal{V}|$. $\mathbf{A}$ is defined as the adjacency matrix, whose element $A(i, j)$ denotes the weight of the edge between vertex i and j, and $A(i,j) = 0$ if there is no edge. $\mathbf{D} = \mathrm{diag}(d_i)$ is the diagonal degree matrix, where $d_i = \sum_j A(i,j)$ is the sum of weights of all edges connected to vertex i. $\mathbf{L} = \mathbf{D} - \mathbf{A}$ defines the Laplacian matrix of the graph, and $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ is the normalized Laplacian matrix, where $\mathbf{I}$ denotes the identity matrix. Both of $\mathbf{L}$ and $\mathcal{L}$ are symmetric positive semidefinite matrices, and have a complete set of orthonormal eigenvectors. We represent the eigenvectors of the normalized Laplacian matrix $\mathcal{L}$ by $\mathbf{u}_i = [u_i(1), u_i(2), \cdots, u_i(N)]^T$ and the associated eigenvalues by λ_i, where $u_i(n)$ is real-valued and $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \leq 2$.

A graph signal is a function defined on the graph and the sample value $f(n)$ at vertex n is represented as a vector $\mathbf{f} = [f(1), f(2), \cdots, f(N)]^T$. The GFT (graph Fourier transform) is then defined as the projections of the graph signal $\mathbf{f}$ onto the eigenvectors of $\mathcal{L}$;

$$F(\lambda_i) = \mathbf{f}^T\mathbf{u_i} = \sum_{n=1}^{N} f(n)u_i(n), \tag{1}$$

and thus the IGFT (inverse graph Fourier transform) is given by

$$f(n) = \mathbf{F}^T\mathbf{U}(n) = \sum_{i=1}^{N} F(\lambda_i)u_i(n), \tag{2}$$

where $\mathbf{F} = [F(\lambda_1), F(\lambda_2), \cdots, F(\lambda_N)]^T$ and $\mathbf{U}(n) = [u_1(n), u_2(n), \cdots, u_N(n)]^T$.

A filtering operation of the graph signal $\mathbf{f}$ in the vertex domain can be expressed in the matrix form as $\mathbf{y} = \mathbf{H}\mathbf{f}$, where the output signal of the filter is $\mathbf{y} = [y(1), y(2), \cdots, y(N)]^T$ and the transform matrix of the filter $\mathbf{H}$ is given by

$$\mathbf{H} = \sum_{i=1}^{N} H(\lambda_i)\mathbf{u}_i\mathbf{u}_i^T, \tag{3}$$

where $H(\lambda)$ is the spectral kernel of the filter.

By using the GFT in Eq.(1) , we have Eq.(4) in the spectral domain;

$$Y(\lambda_i) = H(\lambda_i)F(\lambda_i), \tag{4}$$

where $Y(\lambda_i)$ is the GFT of output signal $\mathbf{y}$.

III. Graph Wavelet Filter Banks

Two channel graph wavelet filter bank $\{\mathbf{H}_k, \mathbf{G}_k\}_{k=0,1}$ proposed in [11] is shown in Fig.1. The corresponding transform matrices are given by

$$\begin{cases} \mathbf{H}_k = \displaystyle\sum_{i=1}^{N} H_k(\lambda_i)\mathbf{u}_i\mathbf{u}_i^T \\ \mathbf{G}_k = \displaystyle\sum_{i=1}^{N} G_k(\lambda_i)\mathbf{u}_i\mathbf{u}_i^T \end{cases}, \tag{5}$$

where $H_k(\lambda), G_k(\lambda)$ are the spectral kernels of analysis and synthesis filters, respectively. $\mathbf{H}_0, \mathbf{G}_0$ act as lowpass filters and $\mathbf{H}_1, \mathbf{G}_1$ are highpass. The down-sampling operation β_L discards the output coefficients of lowpass channel in the set H, while β_H discards the output coefficients of highpass channel in the set L, where $|H| + |L| = N$ and $H \cap L = 0$. Therefore the overall transform matrix is given by

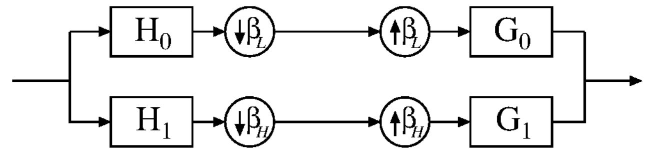

Fig. 1. Two channel graph wavelet filter bank.

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\{\mathbf{G}_0(\mathbf{I} + \mathbf{J}_\beta)\mathbf{H}_0 + \mathbf{G}_1(\mathbf{I} - \mathbf{J}_\beta)\mathbf{H}_1\} \\ &= \frac{1}{2}\{(\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1) + (\mathbf{G}_0\mathbf{J}_\beta\mathbf{H}_0 - \mathbf{G}_1\mathbf{J}_\beta\mathbf{H}_1)\} \end{aligned}, \tag{6}$$

where $\mathbf{J}_\beta = \mathrm{diag}(\beta_i)$ is the diagonal matrix, and β_i is the partition function such that $\beta_i = 1$ if vertex $i \in L$ and $\beta_i = -1$ if vertex $i \in H$. The down-and-up sampling operation β_L in lowpass channel is then expressed in the matrix form as $\frac{1}{2}(\mathbf{I} + \mathbf{J}_\beta)$, while β_H in highpass channel as $\frac{1}{2}(\mathbf{I} - \mathbf{J}_\beta)$. It is known in [11] and [12] that the PR condition of two channel wavelet filter bank is

$$\begin{cases} H_0(\lambda)G_0(\lambda) + H_1(\lambda)G_1(\lambda) = 2 \\ H_0(2-\lambda)G_0(\lambda) - H_1(2-\lambda)G_1(\lambda) = 0 \end{cases}. \tag{7}$$

To cancel the aliasing, the synthesis filters must be chosen as

$$\begin{cases} G_0(\lambda) = H_1(2-\lambda) \\ G_1(\lambda) = H_0(2-\lambda) \end{cases}, \tag{8}$$

therefore the PR condition in Eq.(7) is reduced to

$$H_0(\lambda)H_1(2-\lambda) + H_0(2-\lambda)H_1(\lambda) = 2, \tag{9}$$

which leads to biorthogonal wavelet filter banks.

In addition, from the orthogonality of graph wavelets, it is required to satisfy

$$H_1(\lambda) = H_0(2-\lambda). \tag{10}$$

Therefore, Eq.(9) becomes

$$H_0^2(\lambda) + H_0^2(2-\lambda) = 2. \tag{11}$$

It is clear that only $H_0(\lambda)$ needs to be designed for the orthogonal graph wavelet filter banks (graph QMFs).

IV. Design of Graph QMFs

In this section, we will consider the design of graph QMFs with a polynomial of degree K. It is shown in [11] and [12] that if $H_0(\lambda)$ is a polynomial of degree K, the graph filter is K-hop localized exactly, thus it is capable of implementing the filter iteratively with K one-hop operations at each vertex, without any matrix diagonalization. Therefore, we will discuss the approximation of the desired kernel of the filter with a polynomial of degree K.

$H_0(\lambda)$ is a lowpass filter and has a desired gain $\sqrt{2}$ in passband and 0 in stopband. To ensure $H_0(0) = \sqrt{2}$, we define $H_0(\lambda)$ as

$$H_0(\lambda) = \sqrt{2}(1 + \sum_{k=1}^{K} a_k \lambda^k), \tag{12}$$

where a_k are real-valued coefficients.

If $H_0(\lambda)$ has p zeros at $\lambda = 2$, as in [12] and [13], it can be also written as

$$H_0(\lambda) = \sqrt{2}(1 - \frac{\lambda}{2})^p (1 + \sum_{k=1}^{K-p} b_k \lambda^k), \tag{13}$$

where b_k are real-valued coefficients also. Thus we have

$$\left.\frac{\partial^i H_0(\lambda)}{\partial \lambda^i}\right|_{\lambda=2} = 0 \qquad (i = 0, 1, \cdots, p-1), \tag{14}$$

which means that the filter has a flat spectral at $\lambda = 2$ and the degree of flatness is p.

Theorem 1. *If $H_0(\lambda)$ having p zeros at $\lambda = 2$ satisfies the condition of orthogonality in Eq.(11), then the first $(2p-1)$ filter coefficients in Eq.(12) vanish, i.e., $a_k = 0$ for $k = 1, 2, \cdots, 2p-1$.*

Proof. By differentiating Eq.(11), we have

$$\frac{\partial H_0(\lambda)}{\partial \lambda} = -\frac{H_0(2-\lambda)}{H_0(\lambda)} \frac{\partial H_0(2-\lambda)}{\partial \lambda}. \tag{15}$$

Since $H_0(\lambda)$ has p zeros at $\lambda = 2$, $H_0(2-\lambda)$ includes p zeros at $\lambda = 0$, thus we have

$$\frac{\partial H_0(\lambda)}{\partial \lambda} = \lambda^{2p-1} Q(\lambda), \tag{16}$$

where $Q(\lambda)$ does not include any zero at $\lambda = 0$, i.e., $Q(0) \neq 0$. Therefore, we obtain

$$\left.\frac{\partial^i H_0(\lambda)}{\partial \lambda^i}\right|_{\lambda=0} = 0 \qquad (i = 1, 2, \cdots, 2p-1), \tag{17}$$

which means that it has a flat spectral at $\lambda = 0$ too and the degree of flatness is $2p$. As a result, the first $(2p-1)$ coefficients a_k $(k = 1, 2, \cdots, 2p-1)$ vanish, and Eq.(12) is reduced to

$$H_0(\lambda) = \sqrt{2}(1 + \sum_{k=2p}^{K} a_k \lambda^k). \tag{18}$$

□

A. Maximally Flat Filter

It is proven in Theorem 1 that $H_0(\lambda)$ of graph QMFs is required to have the flat spectral both in passband and stopband if it possesses some zeros at $\lambda = 2$. Now we discuss the design of maximally flat filters whose degrees of flatness are p_1 and p at $\lambda = 0$ and $\lambda = 2$ respectively.

Since it has p zeros at $\lambda = 2$, we have from Eq.(13)

$$H_0(\lambda) = \sqrt{2}(1 + \sum_{n=1}^{p} \binom{p}{n} (-\frac{1}{2})^n \lambda^n)(1 + \sum_{k=1}^{K-p} b_k \lambda^k). \tag{19}$$

In addition, the degree of flatness is p_1 with respect to $\lambda = 0$, then $a_k = 0$ $(k = 1, 2, \cdots, p_1 - 1)$ in Eq.(12). Therefore the minimum degree of the filter is $K = p + p_1 - 1$.

By expanding Eq.(19), we have from the condition of flatness at $\lambda = 0$

$$a_1 = b_1 - \frac{p}{2} = 0 \implies b_1 = \frac{p}{2}, \tag{20}$$

$$a_2 = b_2 - \frac{p}{2} b_1 + \frac{p(p-1)}{8} = 0 \implies b_2 = \frac{p(p+1)}{8}, \tag{21}$$

$$\vdots$$

$$a_k = b_k + \sum_{i=1}^{\min(k,p)} \binom{p}{i} (-\frac{1}{2})^i b_{k-i} = 0 \qquad (k < p_1)$$

$$\implies b_k = -\sum_{i=1}^{\min(k,p)} \binom{p}{i} (-\frac{1}{2})^i b_{k-i}. \tag{22}$$

where $b_0 = 1$. Therefore, the closed-form solution of the maximally flat filter is obtained whose degree is $K = p + p_1 - 1$. The filter coefficients a_k $(k = p_1, p_1 + 1, \cdots, K)$ can be also calculated by

$$a_k = \sum_{i=\max(0,k-p)}^{p_1-1} \binom{p}{k-i} (-\frac{1}{2})^{k-i} b_i. \tag{23}$$

B. Optimization

It is clear from Theorem 1 that it is required to choose $p_1 = 2p$ for the condition of orthogonality. However, it cannot be guaranteed that the maximally flat filters only with $p_1 = 2p$ satisfy the condition of orthogonality. It is because the condition of flatness in Eq.(17) is only the necessary condition for the orthogonality in Theorem 1. It is insufficient to use only the condition of flatness in the design of $H_0(\lambda)$. It is known in [11] and [12] that the polynomial approximation of the desired kernel cannot be exactly orthogonal without any reconstruction error. Therefore, it is necessary to minimize the reconstruction error of graph QMFs. We define the reconstruction error function $E(\lambda)$ as

$$E(\lambda) = H_0^2(\lambda) + H_0^2(2-\lambda) - 2. \tag{24}$$

The goal is to minimize the maximum of $|E(\lambda)|$ within $0 \leq \lambda \leq 2$ by increasing the polynomial degree of the kernel filter.

Theorem 2. *If $H_0(\lambda)$ of degree $K \geq p + p_1$ has the form in Eq.(13) and the first $(p_1 - 1)$ coefficients b_k are given in Eqs.(20)∼(22), then the filter has a flat spectral with the degrees of flatness p_1 and p at $\lambda = 0$ and $\lambda = 2$ respectively, regardless of what the remaining coefficients b_k $(k \geq p_1)$ are.*

Proof. The proof is straightforward. Since $H_0(\lambda)$ has the form in Eq.(13), it has p zeros at $\lambda = 2$ and satisfies the condition of flatness in Eq.(14). Furthermore, the first $(p_1 - 1)$ coefficients b_k are given in Eqs.(20)∼(22), then the first (p_1-1) filter coefficients a_k in Eq.(12) vanish, i.e., $a_k = 0$ for $k = 1, 2, \cdots, p_1 - 1$. Thus, the condition of flatness at $\lambda = 0$ with the degree of flatness p_1 is satisfied. □

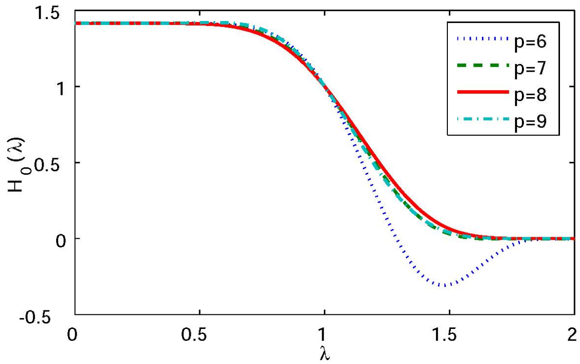

Fig. 2. Spectral responses of $H_0(\lambda)$ in Example 1.

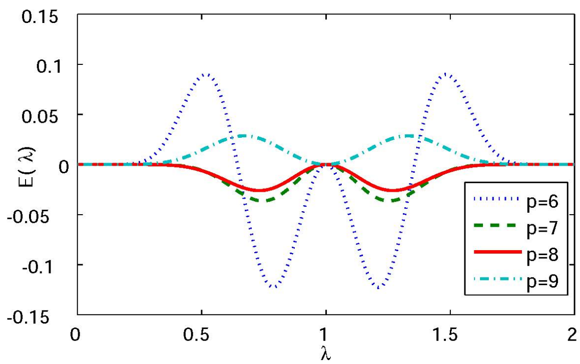

Fig. 3. Reconstruction error $E(\lambda)$ in Example 1.

It is known from Theorem 2 that the coefficients b_k $(k = 1, 2, \cdots, p_1 - 1)$ in Eq.(13) can be obtained from the maximally flat filters by using Eqs.(20)∼(22), which guarantees the flat spectral of $H_0(\lambda)$ both in passband and stopband. Therefore, we can use the remaining coefficients b_k $(p_1 \leq k \leq K - p)$ to optimize $H_0(\lambda)$, without any influence on the degree of flatness at $\lambda = 0$ and $\lambda = 2$. In this paper, we want to minimize the reconstruction error $E(\lambda)$ by using the remaining degree of freedom. A simplest example is the filter of degree $K = p + p_1$. The only remaining coefficient is b_{p_1} that is used to ensure $H_0(1) = 1$. Thus we have from Eq.(13)

$$b_{p_1} = 2^{p-\frac{1}{2}} - 1 - \sum_{k=1}^{p_1-1} b_k. \tag{25}$$

Example 1: We have designed $H_0(\lambda)$ of degree $K = p + p_1 = 18$ with $\{p, p_1\} = \{6, 12\}, \{7, 11\}, \{8, 10\}, \{9, 9\}$. The resulting spectral responses of $H_0(\lambda)$ are shown in Fig. 2, and the reconstruction errors $E(\lambda)$ in Fig. 3. It is seen in Fig. 2 that the spectral responses are monotonously decreasing except $p = 6$. The reconstruction error is 0 at $\lambda = 1$ and varies with a different choice of $\{p, p_1\}$, as shown in Fig. 3. It is found that if $p \leq p_1 \leq 2p$ is chosen, we can obtain a reasonable reconstruction error. In this example, the filter of $p = 8$ has the minimum reconstruction error.

It is seen in *Example 1* that we used the only remaining degree of freedom to reduce the reconstruction error. However, the error is still large. To reduce further the reconstruction error, it is necessary to increase the filter degree, that is, $K > p + p_1$. In [14], Tay and Lin had also discussed the design of graph QMFs with the flatness constraints in passband and stopband, where only one coefficient was used to reduce the reconstruction error in addition to $H_0(1) = 1$, that is, the filter degree is fixed to $K = p + p_1 + 1$.

In the following, we discuss how to minimize the reconstruction error of graph QMFs by using the remaining degrees of freedom b_k $(p_1 \leq k \leq K - p)$ in Eq.(13), i.e.,

$$\min_{\mathbf{b}} \{ \max_{0 \leq \lambda_i \leq 2} |E(\lambda_i)| \}, \tag{26}$$

where $\mathbf{b} = [b_{p_1}, b_{p_1+1}, \cdots, b_{K-p}]$. Note that the filter degree K can be arbitrarily chosen in this paper.

Let δ be the maximum error, then we have

$$-\delta < E(\lambda_i) = H_0^2(\lambda_i) + H_0^2(2 - \lambda_i) - 2 < \delta. \tag{27}$$

Therefore, the optimization problem is stated as

$$\min_{\mathbf{b}} \delta$$
$$\text{subject to} \begin{cases} H_0^2(\lambda_i) + H_0^2(2 - \lambda_i) - \delta - 2 < 0 \\ -H_0^2(\lambda_i) - H_0^2(2 - \lambda_i) - \delta + 2 < 0 \end{cases} \tag{28}$$

where if the graph has been given, then λ_i $(i = 1, 2, \cdots, N)$ are known. Thus we can evaluate the error $E(\lambda)$ only at λ_i, however, the obtained filter is useful only for graphs with the same structure. For the graph-independent case, all of λ in $[0, 1]$ need to be considered, since $E(\lambda) = E(2 - \lambda)$. We use a discretization of $\lambda_i = \frac{i}{L}$ $(i = 0, 1, \cdots, L)$, where L is the number of the points to be evaluated.

Moreover, if a sharper stopband is required as discussed in [13], it is also possible to impose some zeros λ_m $(m = 0, 1, \cdots, M)$ in the stopband;

$$H_0(\lambda_m) = \sqrt{2}(1 - \frac{\lambda_m}{2})^p (1 + \sum_{k=1}^{K-p} b_k \lambda_m^k) = 0. \tag{29}$$

Since $1 < \lambda_m < 2$, Eq.(29) can be reduced to

$$1 + \sum_{k=1}^{K-p} b_k \lambda_m^k = 0 \tag{30}$$

which are added into the optimization problem in Eq.(28).

It is possible to solve the optimization problem in Eq.(28) by using the *fmincon* function in Matlab. The *fmincon* function needs a good initial solution. It is seen in *Example 1* that the b_{p_1} in Eq.(25) has yielded a reasonable solution. Thus we use the b_{p_1} in Eq.(25) as the initial solution $\mathbf{b}^0 = [b_{p_1}, 0, \cdots, 0]$. The number of unknown coefficients b_k $(p_1 \leq k \leq K - p)$ is $K - p - p_1 + 1$, whereas $K - p$ in [13], that is, b_k $(1 \leq k \leq K - p)$. Therefore, the proposed design method is more computationally efficient.

Example 2: We firstly used $H_0(\lambda)$ designed in *Example 1* as the initial solution, whose degree is $K = p + p_1 = 18$

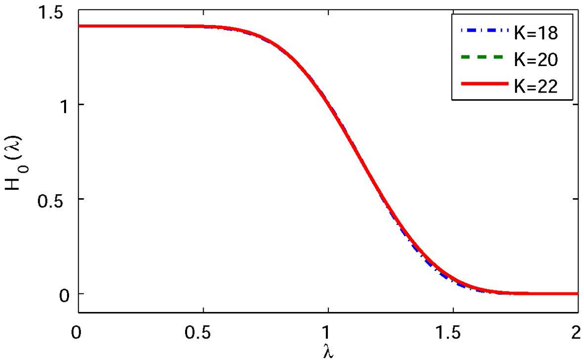

Fig. 4. Spectral responses of $H_0(\lambda)$ in Example2.

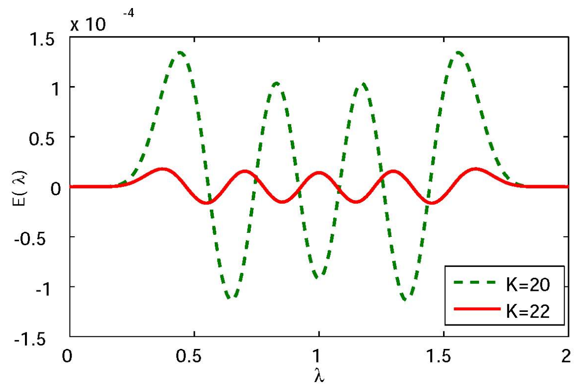

Fig. 5. Reconstruction error $E(\lambda)$ in Example 2.

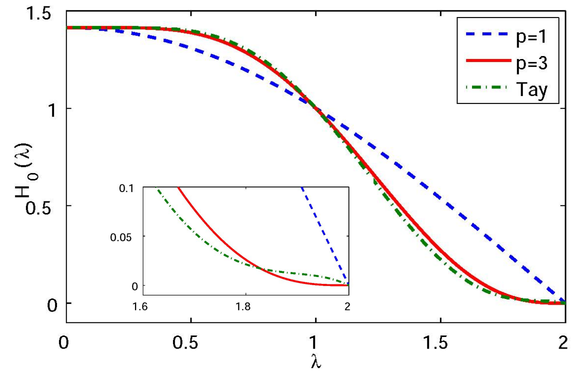

Fig. 6. Spectral responses of $H_0(\lambda)$ in Example 3.

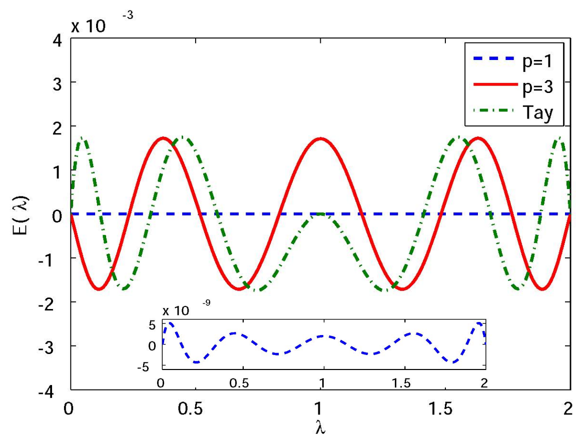

Fig. 7. Reconstruction error $E(\lambda)$ in Example 3.

with $p = 8, p_1 = 10$. By increasing the filter degree to $K = 20$, we have designed $H_0(\lambda)$ with the same degree of flatness to minimize the reconstruction error. The resulting spectral response is shown in dotted line in Fig. 4, and the reconstruction error $E(\lambda)$ in Fig. 5. The spectral response of the initial solution with $K = 18$ is also shown in dashed line in Fig. 4 for comparison. The maximum reconstruction error of the initial solution is $E_{max} = 0.01972$, while it has been reduced to $E_{max} = 0.0001344$ with the filter of $K = 20$. The spectral response and reconstruction error of $H_0(\lambda)$ of degree $K = 22$ are also shown in solid line in Fig. 4 and Fig. 5, respectively. It is seen in Fig. 4 that these filters have almost the same spectral. The maximum reconstruction error is further reduced to $E_{max} = 0.00001783$ in Fig. 5.

Example 3: We have designed $H_0(\lambda)$ of degree $K = 6$ with $p = p_1 = 1$, which is the same as *Example 1* in [13]. Note that $p_1 = 1$ means the filter without any constraint of flatness at $\lambda = 0$, and the method proposed in [13] cannot design the filter with the flatness constraint at $\lambda = 0$. The spectral of $H_0(\lambda)$ is shown in dotted line in Fig. 6, and the reconstruction error $E(\lambda)$ in Fig. 7. For comparison, the results designed by Tay and Lin are also shown in dashed line in Fig. 6 and Fig. 7, respectively. It is seen that the Tay-Lin's filter has a sharper spectral response than the proposed filter in Fig. 6, however, the reconstruction error of our filter ($E_{max} = 5.094 \times 10^{-9}$) is much smaller than their filter ($E_{max} = 0.001747$) in Fig. 7. It is because their initial solution is generated from the Bernstein polynomial and different from that in this paper. From the viewpoint of minimizing the maximum reconstruction error while satisfying the flatness constraints, our solution is better than their filter. To obtain a sharper stopband, we have increased the degree of flatness at $\lambda = 2$ to $p = 3$. The resulting spectral and reconstruction error are also shown in solid line in Fig. 6 and Fig. 7. It is clear that the spectral of this filter is close to the Tay-Lin's filter, and the maximum reconstruction error is $E_{max} = 0.001721$ which is slightly smaller. It should be noted that the Tay-Lin's filter has only one zero ($p = 1$) at $\lambda = 2$, while the filter of $p = 3$ has three zeros and is flatter at $\lambda = 2$.

V. Application

To demonstrate that the proposed graph wavelet QMFs is useful in analyzing and compressing arbitrary signals defined on irregular graphs, we have applied the kernel filters of degree $K = 6$ in *Example 3* to the Minnesota traffic graph signal from [10], [11], [12]. The graph is shown in Fig. 8(a), and the graph signal in Fig. 8(b), where the color of the node represents the sample value. We used the Matlab code provided in [11] and

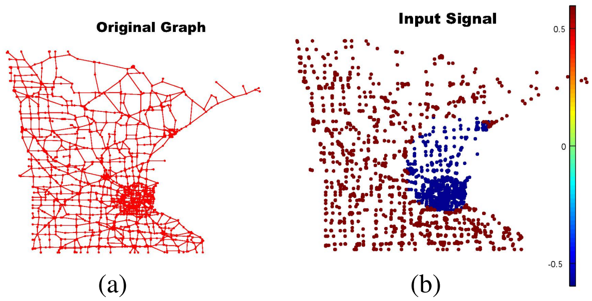

Fig. 8. (a) Minnesota traffic graph, and (b) the graph signal.

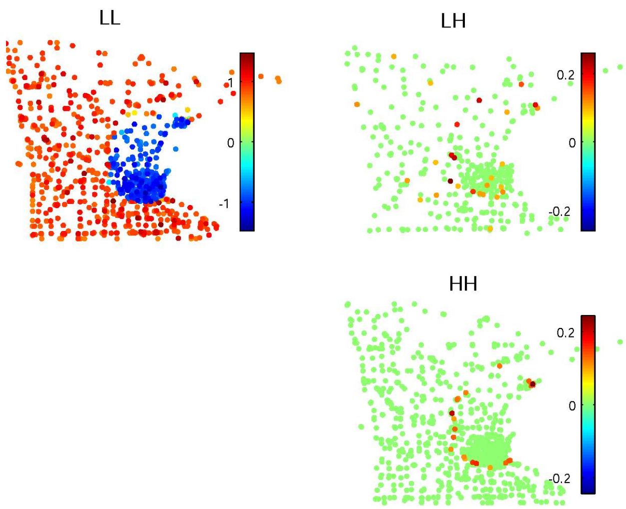

Fig. 9. Subband coefficients of the filter bank with $p = p_1 = 1$.

[12] to implement it. The subband coefficients obtained by using the filter bank with $p = p_1 = 1$ in *Example 3* are shown in Fig. 9. Similarly as in [11], [12] and [13], the LL channel is an approximation of the original signal, while the LH and HH channels capture highpass details for reconstruction. Note that the HL channel is empty after downsampling because the graph is perfectly 3-colorable.

The signal is reconstructed by using γ (e.g., 1%) absolute-largest highpass coefficients in the LH and HH channels and all of lowpass coefficients in the LL channel. The compression performance has been investigated, and the Signal-to-Noise-Ratio (SNR) of the reconstructed graph signal is given in Table I. For comparison, the results obtained from the Tay-Lin's kernel filter in *Example 1* of [13] and the Meyer kernel approximation with the same length in [11] are also given in Table I. It is seen that the proposed graph QMFs outperform the existing graph wavelet filter banks. In particular, the kernel filter with $p = p_1 = 1$ has significantly reduced the reconstruction error.

TABLE I
SNR(dB) OF RECONSTRUCTED GRAPH SIGNAL

γ	$p = 1$	$p = 3$	Tay-Lin	Meyer
0.01	**14.20**	13.93	13.91	12.43
0.02	**14.84**	14.52	14.49	12.90
0.03	**15.35**	15.02	14.99	13.29
0.05	**16.26**	15.90	15.86	14.04
0.08	**17.52**	17.11	17.07	15.10
0.10	**18.37**	17.91	17.86	15.82
0.20	**22.66**	21.82	21.73	19.31
0.30	**27.13**	25.98	25.87	22.89
0.50	**47.15**	44.83	44.68	28.28
1.00	**180.34**	64.43	72.99	28.37

VI. CONCLUSION

In this paper, we have proposed a new design method of two channel compactly supported near orthogonal graph wavelet filter banks. Firstly, we have proven the necessary condition for the orthogonality where the kernel filter must have a flat spectral both in passband and stopband. we then have derived the closed-form solution of the maximally flat filters. From the maximally flat filters, we have given a kernel filter function that ensures the specified degrees of flatness at $\lambda = 0$ and $\lambda = 2$, thus we can use the remaining coefficients to minimize the reconstruction error. The proposed design method is computationally efficient. Several examples have been designed and compared with the method proposed in [13]. Finally, the proposed graph wavelets have been applied to the Minnesota traffic graph to demonstrate the effectiveness.

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