Closed-Form Design of Maximally Flat IIR Half-Band Filters

Xi Zhang, *Senior Member, IEEE*, and Kevin Amaratunga

***Abstract*—Half-band (HB) filters are of great importance and are often used in multirate digital signal processing systems, filter banks and wavelets. In this paper, a new closed-form expression for the transfer function of the maximally flat (MF) infinite-impulse response (IIR) HB filters is presented. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the maximal flatness conditions. The proposed IIR half-band filters are more general than the existing half-band filters, because they include the conventional finite-impulse response (FIR) half-band filters with exactly linear phase, the generalized FIR half-band filters with approximately linear phase and the all-pass-based IIR half-band filters, as special cases. Furthermore, the causal stable IIR HB filters and the IIR HB filters with exactly linear phase can be realized also. Finally, some design examples are presented to demonstrate the effectiveness of the proposed IIR HB filters.**

***Index Terms*—Closed-form solution, half-band filter, infinite-impulse response (IIR) digital filter, maximally flat response.**

I. Introduction

HALF-BAND (HB) filters are an important class of digital filters and are often used in multirate digital signal processing systems, filter banks and wavelets [1]–[5]. For example, it is well-known [3], [4], [9], [10], [12] that the design problem of two-band perfect reconstruction filter banks including orthonormal and biorthogonal filter banks can be reduced to that of HB filters. Therefore, it is very important how to design HB filters according to the given specification. In many applications of filter banks and wavelets, such as wavelet-based image coding, HB filters are required to possess the maximally flat (MF) frequency response to get better coding performance [3], [4]. Much work has been done until now, which is mainly devoted to the design of finite-impulse response (FIR) HB filters [1]–[8], [15]–[21]. The closed-form solution for the MF FIR HB filters with exactly linear phase can be found in [1]–[4] and [18], while that for the generalized MF FIR HB filters with approximately linear phase is recently presented in [15] and [19]–[21]. In contrast, there exists little work regarding infinite-impulse response (IIR) HB filters [12], [13], [22]. It is known that IIR filters tend to be of much lower order for meeting the same specification as compared with FIR filters. A class of IIR HB filters has been given in [2]–[4], [10], [12] by using the parallel structure of a pure delay section and an all-pass subfilter and applied in the design of filter banks and wavelets. It has been shown also in [10] and [12] that the design of such all-pass-based IIR HB filters is equivalent to the phase approximation of the all-pass subfilter. A detailed review about the all-pass filter design has been given in [11]. The closed-form solution for the all-pass-based MF IIR HB filters can be found in [11] and [12]. The MF IIR HB filters with exactly linear phase are also given as a special case of the generalized Butterworth filters proposed in [13].

In this paper, we propose a more general class of IIR HB filters than the existing HB filters. The proposed IIR HB filters include not only the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase and the all-pass-based IIR HB filters as special cases, but also the causal stable IIR HB filters and the IIR HB filters with exactly linear phase. The IIR HB filters with exactly linear phase are generally needed in image processing applications and have a better magnitude response than the FIR counterparts, although the resulting filters are noncausal. The noncausal IIR filters, however, can be realized by dividing them into the causal and anticausal stable parts to implement in some applications such as image processing and offline processing. We present a new closed-form expression for the transfer function of the MF IIR HB filters. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations that are derived from the maximal flatness conditions. We also investigate the conditions for realizing the causal stable IIR HB filters and the IIR HB filters with exactly linear phase and show that the MF IIR HB filters with exactly linear phase proposed in [13] are only a subclass of that proposed in this paper. Finally, some design examples are presented and compared with the existing MF HB filters to demonstrate the effectiveness of the proposed MF IIR HB filters.

This paper is organized as follows. Section II derives the transfer function from the time-domain condition for general IIR HB filters and examines the relation with the existing HB filters. Some filter properties and the flatness condition are shown in Section III. Section IV gives a new closed-form expression for the MF IIR HB filters. The relationship between the proposed MF IIR HB filters and the existing MF HB filters is discussed in Section V. Section VI presents some design examples and a comparison with the existing MF HB filters.

Manuscript received May 25, 2001 revised July 25, 2002. This paper was recommended by Associate Editor L. Yong.

X. Zhang is with the Department of Electrical Engineering, Nagaoka University of Technology, Niigata 940-2188, Japan (e-mail: xiz@nagaokaut.ac.jp).

K. Amaratunga is with the Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: kevina@mit.edu).

Digital Object Identifier 10.1109/TCSII.2002.803469.

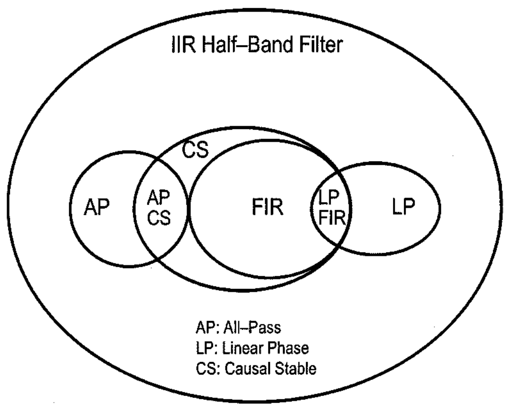

Fig. 1. IIR Half-Band filters. All-pass filter (AP). Causal-stable filter (CS). Linear-phase filter (LP).

II. IIR Half-Band Filters

Let $h_n \;\; (n = 0, 1, \ldots)$ be the impulse response of an IIR HB filter. It is well-known that the impulse response h_n should satisfy the following condition:

$$\begin{cases} h_K = \frac{1}{2} \\ h_{K+2k} = 0, \quad (k = \pm 1, \pm 2, \ldots) \end{cases} \tag{1}$$

where K is the desired group delay of the HB filter. In genaral, K is odd number. Note that if K is an even number, then we have $h_0 = 0$ from (1). The HB filter is composed of a delay element z^{-1} and a HB filter with an odd group delay of $K-1$. In the following, we will consider the design of HB filters with an odd K. According to the time-domain condition in (1), the transfer function $H(z)$ of the IIR HB filter can be given by

$$H(z) = \frac{1}{2} z^{-K} + G\left(z^2\right). \tag{2}$$

If $G(z)$ is FIR filter, then $H(z)$ becomes FIR HB filter. In this paper, we assume that $G(z)$ is a general IIR filter with numerator degree N and denominator degree M

$$G(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}} \tag{3}$$

where a_n and b_m are real coefficients and $b_0 = 1$. It should be noted that if $M = 0$, $G(z)$ will degenerate into FIR filter. Then $H(z)$ becomes the generalized FIR HB filters proposed in [15] and [19]–[21]. If we further impose the constraints of $N = K$ and $a_n = a_{N-n}$ on $G(z)$, then $H(z)$ will be the conventional FIR HB filters with exactly linear phase. In addition, if $N = M$ and $b_n = Ca_{N-n}$, where C is constant, then $G(z)$ will be allpass with gain $1/C$ and $H(z)$ is the all-pass-based IIR HB filters presented in [2], [10], and [12], which are based on the parallel structure of a pure delay section z^{-K} and an all-pass subfilter. To summarize, as shown in Fig. 1, the IIR HB filters proposed in this paper include the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters and the all-pass-based IIR HB filters as special cases, therefore, are more general than the existing HB filters. Furthermore, we can get the causal stable IIR HB filters by constraining its poles inside the unit circle and the IIR HB filters with exactly linear phase by imposing the constraints of $K = N - M$ and $a_n = a_{N-n}, b_m = b_{M-m}$ on $G(z)$ when N is odd and M even. The IIR HB filters with exactly linear phase are noncausal, but needed in image processing applications, in general. The noncausal IIR filters can be divided into the causal and anticausal stable parts that have the poles inside and outside the unit circle respectively and can be realized in some applications such as image processing and offline processing.

III. Filter Properties

In general, HB filters are required to be lowpass with the desired group delay of K. The desired frequency response of $H(z)$ is given by

$$H_d\left(e^{j\omega}\right) = \begin{cases} e^{-jK\omega}, & (0 \le \omega \le \omega_p) \\ 0, & (\omega_s \le \omega \le \pi) \end{cases} \tag{4}$$

where ω_p, ω_s are the cutoff frequencies of the passband and stopband, respectively and $\omega_p + \omega_s = \pi$. From (2), we have

$$G_d\left(e^{j2\omega}\right) = \begin{cases} \frac{1}{2} e^{-jK\omega}, & (0 \le \omega \le \omega_p) \\ -\frac{1}{2} e^{-jK\omega}, & (\omega_s \le \omega \le \pi) \end{cases} \tag{5}$$

that is

$$G_d\left(e^{j\omega}\right) = \frac{1}{2} e^{-j(K/2)\omega}, \quad (0 \le \omega \le 2\omega_p) \tag{6}$$

which means that $G(z)$ is required to be allpass with gain 1/2 in the band [0, $2\omega_p$]. Let $\hat{H}(z)$ be the advanced version of $H(z)$

$$\hat{H}(z) = z^K H(z) = \frac{1}{2} + \frac{\sum_{n=0}^{N} a_n z^{K-2n}}{\sum_{m=0}^{M} b_m z^{-2m}} \tag{7}$$

then there exist the following relations between $H(z)$ and $\hat{H}(z)$

$$\begin{cases} \left|H\left(e^{j\omega}\right)\right| = \left|\hat{H}\left(e^{j\omega}\right)\right| \\ \theta(\omega) = \hat{\theta}(\omega) - K\omega \\ \tau(\omega) = \hat{\tau}(\omega) + K \end{cases} \tag{8}$$

where $|H(e^{j\omega})|$, $\theta(\omega)$, $\tau(\omega)$ and $|\hat{H}(e^{j\omega})|$, $\hat{\theta}(\omega)$, $\hat{\tau}(\omega)$ are the magnitude, phase and group delay responses of $H(z)$ and $\hat{H}(z)$, respectively. Therefore, the desired frequency response of $\hat{H}(z)$ is

$$\hat{H}_d\left(e^{j\omega}\right) = \begin{cases} 1, & (0 \le \omega \le \omega_p) \\ 0, & (\omega_s \le \omega \le \pi) \end{cases}. \tag{9}$$

The frequency response of $\hat{H}(z)$ is given from (7) by

$$\hat{H}\left(e^{j\omega}\right) = \frac{1}{2} + \frac{\sum_{n=0}^{N} a_n e^{j(K-2n)\omega}}{\sum_{m=0}^{M} b_m e^{-j2m\omega}} \tag{10}$$

and satisfies

$$\hat{H}\left(e^{j\omega}\right) + \hat{H}^*\left(e^{j(\pi-\omega)}\right) \equiv 1 \tag{11}$$

where x^* denotes the complex conjugate of x. This means that if $\hat{H}(e^{j\omega}) = 0$ in the stopband, then the frequency response of $\hat{H}(z)$ becomes 1 in the passband. Therefore, we just need to approximate the stopband response in the HB filter design.

In many applications such as the wavelet-based image coding, HB filters are required to be maximally flat (MF). The flatness condition is given by

$$\left.\frac{\partial^i |H\left(e^{j\omega}\right)|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, N+M). \tag{12}$$

From (8), we get

$$\left.\frac{\partial^i |\hat{H}\left(e^{j\omega}\right)|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, N+M). \tag{13}$$

Since $\hat{H}(e^{j\omega}) = |\hat{H}(e^{j\omega})| e^{j\hat{\theta}(\omega)}$, then we have

$$\frac{\partial^i \hat{H}\left(e^{j\omega}\right)}{\partial \omega^i} = \sum_{n=0}^{i} \binom{i}{n} \frac{\partial^n \left|\hat{H}\left(e^{j\omega}\right)\right|}{\partial \omega^n} \frac{\partial^{i-n}\left\{e^{j\hat{\theta}(\omega)}\right\}}{\partial \omega^{i-n}}. \tag{14}$$

Therefore, it can be derived from (14) that (13) is equivalent to

$$\left.\frac{\partial^i \hat{H}\left(e^{j\omega}\right)}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, N+M). \tag{15}$$

That is, (12) is equivalent to (15). This means that $H(z)$ must have $N+M+1$ zeros located at $z = -1$. It can be derived from (11) that if the flatness condition in (15) is satisfied, then $\hat{H}(z)$ will satisfy also

$$\begin{cases} \hat{H}(1) = 1 \\ \left.\frac{\partial^i \hat{H}\left(e^{j\omega}\right)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, N+M) \end{cases} \tag{16}$$

that is

$$\begin{cases} \hat{H}_R(0) = 1 \\ \left.\frac{\partial^i \hat{H}_R(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, N+M) \end{cases} \tag{17}$$

and

$$\left.\frac{\partial^i \hat{H}_I(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \ldots, N+M) \tag{18}$$

where $\hat{H}_R(\omega) = \Re\{\hat{H}(e^{j\omega})\}$, $\hat{H}_I(\omega) = \Im\{\hat{H}(e^{j\omega})\}$ are the real and imaginary parts of $\hat{H}(e^{j\omega})$ respectively, i.e., $\hat{H}(e^{j\omega}) = \hat{H}_R(\omega) + j\hat{H}_I(\omega)$. The magnitude, phase and group delay responses of $\hat{H}(z)$ can be expressed in terms of $\hat{H}_R(\omega)$, $\hat{H}_I(\omega)$ as

$$\begin{cases} \left|\hat{H}\left(e^{j\omega}\right)\right| = \sqrt{\hat{H}_R(\omega)^2 + \hat{H}_I(\omega)^2} \\ \hat{\theta}(\omega) = \tan^{-1} \frac{\hat{H}_I(\omega)}{\hat{H}_R(\omega)} \\ \hat{\tau}(\omega) = -\frac{\partial \hat{\theta}(\omega)}{\partial \omega} = \frac{\hat{H}'_R(\omega)\hat{H}_I(\omega) - \hat{H}_R(\omega)\hat{H}'_I(\omega)}{\hat{H}_R(\omega)^2 + \hat{H}_I(\omega)^2} \end{cases}. \tag{19}$$

Hence, it is clear from (17) and (18) that the magnitude, phase and group delay responses of $\hat{H}(z)$ satisfy

$$\begin{cases} |\hat{H}(1)| = 1 \\ \left.\frac{\partial^i |\hat{H}\left(e^{j\omega}\right)|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, N+M) \end{cases} \tag{20}$$

and

$$\begin{cases} \left.\frac{\partial^i \hat{\theta}(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \ldots, N+M) \\ \left.\frac{\partial^i \hat{\tau}(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 0, 1, \ldots, N+M-1) \end{cases}. \tag{21}$$

That is, the magnitude and group delay responses of $H(z)$ satisfy

$$\begin{cases} |H(1)| = 1 \\ \left.\frac{\partial^i |H\left(e^{j\omega}\right)|}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, N+M) \end{cases} \tag{22}$$

and

$$\begin{cases} \tau(0) = K \\ \left.\frac{\partial^i \tau(\omega)}{\partial \omega^i}\right|_{\omega=0} = 0 \quad (i = 1, 2, \ldots, N+M-1) \end{cases} \tag{23}$$

which mean that $H(z)$ has both the maximally flat magnitude and group delay responses at $\omega = 0$ also, if the flatness condition in (12) is satisfied at $\omega = \pi$. Therefore, we just need to consider the flatness condition in (12) in the MF HB filter design.

IV. Closed-Form Solution for MF IIR HB Filters

In this section, we derive a new closed-form expression for the MF IIR HB filters from the flatness condition in (12). We have from (10)

$$\hat{H}\left(e^{j\omega}\right) = \frac{N(\omega)}{D(\omega)} \tag{24}$$

where

$$\begin{cases} N(\omega) = \sum_{n=0}^{N} a_n e^{j(K-2n)\omega} + \frac{1}{2} \sum_{m=0}^{M} b_m e^{-j2m\omega} \\ D(\omega) = \sum_{m=0}^{M} b_m e^{-j2m\omega} \end{cases}. \tag{25}$$

Since $H(z)$ must have $N+M+1$ zeros located at $z = -1$, the flatness condition in (12) is equivalent to

$$\left.\frac{\partial^i N(\omega)}{\partial \omega^i}\right|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, N+M). \tag{26}$$

From (25), we have

$$\left.\frac{\partial^i N(\omega)}{\partial \omega^i}\right|_{\omega=\pi} = -\sum_{n=0}^{N} a_n\{j(K-2n)\}^i + \frac{1}{2}\sum_{m=0}^{M} b_m(-j2m)^i. \quad (27)$$

Substituting (27) into (26), we get

$$2\sum_{n=0}^{N} a_n(K-2n)^i - \sum_{m=0}^{M} b_m(-2m)^i = 0 \; (i=0,1,\ldots,N+M). \quad (28)$$

Since $b_0 = 1$, we have (29) at the bottom of the page. We rewrite (29) in matrix form as

$$\boldsymbol{VDa} = \boldsymbol{u} \quad (30)$$

where $\boldsymbol{a} = [a_0, a_1, \ldots, a_N, b_1, \ldots, b_M]^T$, $\boldsymbol{u} = [1, 0, \ldots, 0]^T$: see (31) at the bottom of the page, and $\boldsymbol{D} = \mathrm{diag}[d_0, d_1, \ldots, d_{N+M}]$ with

$$d_i = \begin{cases} 2 & (0 \le i \le N) \\ -1 & (N+1 \le i \le N+M) \end{cases}. \quad (32)$$

It should be noted that $\boldsymbol{V}$ is the Vandermonde matrix with distinct elements since K is odd number. Therefore, there is always a unique solution. By using the Cramer's rule and Vandermonde determinant [23], a closed-form solution can be obtained by

$$\begin{cases} a_n = \frac{(-1)^{N-n}}{2} \frac{M!}{n!(N-n)!} \frac{\prod_{i=0}^{N}\left(\frac{K}{2}-i\right)}{\prod_{i=0}^{M}\left(\frac{K}{2}+i-n\right)} \\ b_m = (-1)^m \binom{M}{m} \prod_{i=0}^{N} \frac{\frac{K}{2}-i}{\frac{K}{2}-i+m} \end{cases}. \quad (33)$$

V. Relation With the Existing MF HB Filters

In this section, we examine the relationship between the proposed MF IIR HB filters and the existing MF HB filters and present some new MF IIR HB filters, such as the exactly linear phase IIR HB filters and the causal stable IIR HB filters.

A. Generalized MF FIR HB Filters

When $M = 0$, $G(z)$ is FIR filter and then $H(z)$ becomes FIR HB filter. In this case, the closed-form solution for the MF HB filters can be obtained by substituting $M = 0$ into (33), that is

$$a_n = \frac{(-1)^n}{(2n-K)} \frac{\prod_{i=0}^{N}\left(i - \frac{K}{2}\right)}{n!(N-n)!} \quad (34)$$

which is the same as that for the generalized MF FIR HB filters given in [15], [19]–[21]. The generalized MF FIR HB filters have an approximately linear phase response in the passband.

B. Conventional MF FIR HB Filters With Exactly Linear Phase

Let $N = K$, then (34) becomes

$$a_n = \frac{(-1)^{(N+1)/2+n}}{(2n-N)} \frac{\prod_{i=0}^{(N-1)/2}\left(i+\frac{1}{2}\right)^2}{n!(N-n)!} \quad (35)$$

where N is an odd number since K is odd. It is easy to verify that the linear phase condition of $a_n = a_{N-n}$ is satisfied. Therefore, the MF FIR HB filters have an exactly linear phase response.

C. All-Pass-Based IIR HB Filters

If we assume that $N = M$, then (33) becomes (36) at the bottom of the page. It is clear that the condition of $b_n = 2a_{N-n}$ is satisfied. Therefore, $G(z)$ is an all-pass filter with gain 1/2

$$\begin{cases} 2\sum_{n=0}^{N} a_n - \sum_{m=1}^{M} b_m = 1 \\ 2\sum_{n=0}^{N} a_n(K-2n)^i - \sum_{m=1}^{M} b_m(-2m)^i = 0 \quad (i=1,2,\ldots,N+M) \end{cases}. \quad (29)$$

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ K & K-2 & \cdots & K-2N & -2 & \cdots & -2M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ K^{N+M} & (K-2)^{N+M} & \cdots & (K-2N)^{N+M} & (-2)^{N+M} & \cdots & (-2M)^{N+M} \end{bmatrix} \quad (31)$$

$$\begin{cases} a_n = \frac{(-1)^{N-n}}{2} \binom{N}{n} \prod_{i=0}^{N} \frac{\frac{K}{2}-i}{\frac{K}{2}+i-n} = \frac{1}{2}\binom{N}{n} \prod_{i=1}^{N-n} \frac{N+1-\frac{K}{2}-i}{\frac{K}{2}+i} \\ b_n = (-1)^n \binom{N}{n} \prod_{i=0}^{N} \frac{\frac{K}{2}-i}{\frac{K}{2}-i+n} = \binom{N}{n} \prod_{i=1}^{n} \frac{N+1-\frac{K}{2}-i}{\frac{K}{2}+i}. \end{cases} \quad (36)$$

TABLE I
RANGE OF K FOR THE CAUSAL STABLE IIR HB FILTERS

$M \backslash N$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
2	1	3	5	5	7	7	9	11	11	13	13	15	15	17	19
3	3	3	5	7	7	9	11	11	13	15	15	17	17	19	21
4	3	5	5	7	9	9	11	13	13	15	17	17	19	21	21
5	3	5	5	7	9	11	11	13	15	17	17	19	21	21	23
6	3	5	7	7	9	11	13	15	15	17	19	19	21	23	25
7	[3]	5	7	9	9	11	13	15	17	17	19	21	23	23	25
8	x	[5, 9]	7	9	11	11	13	15	17	19	19	21	23	25	25
9	x	[5]	7	9	11	11	13	15	17	19	21	21	23	25	27
10	x	x	[7, 9]	9	11	13	13	15	17	19	21	23	23	25	27
11	x	x	[7]	[9, 15]	11	13	15	15	17	19	21	23	25	25	27
12	x	x	x	[9]	11	13	15	17	17	19	21	23	25	27	27
13	x	x	x	x	[11, 13]	13	15	17	19	19	21	23	25	27	29
14	x	x	x	x	[11]	[13, 17]	15	17	19	21	21	23	25	27	29
15	x	x	x	x	x	[13]	[15, 21]	17	19	21	23	23	25	27	29

and $H(z)$ is the all-pass-based IIR HB filter proposed in [2], [10], and [12]. As shown in [2], [10], and [12], the all-pass-based IIR HB filters may not be causal stable depending on the group delay K. To get the causal stable IIR HB filters, all of their poles must be located inside the unit circle. It is known in [14] that the causal stable all-pass filter of order N has a monotonically decreasing phase response and its phase is $-N\pi$ at $\omega = \pi$. If k poles of the all-pass filter are located outside the unit circle, then its phase is $-(N-2k)\pi$ at $\omega = \pi$. For example, when there is only 1 pole outside the unit circle, its phase is $-(N-2)\pi$ at $\omega = \pi$. It is seen from (6) that the desired phase response of $G(z)$ is $-(K/2)\omega$. To ensure the obtained HB filters are causal stable, the phase response of $G(z)$ must be closer to $-N\pi$ rather than $-(N-2)\pi$ when ω approaches π. Therefore, K should be chosen to satisfy

$$K > 2(N-1). \tag{37}$$

Since K is odd number, the minimal group delay for the causal stable IIR HB filters is $K_{\min} = 2N - 1$.

D. Causal Stable IIR HB Filters

Like the all-pass-based IIR HB filters, the proposed IIR HB filters may not be causal stable depending on the group delay K. To get a causal stable IIR HB filter, we have to choose K carefully. For example, in the case of $M = 1$, $G(z)$ has just one pole of $z_p = (K-2N)/(K+2)$. For $G(z)$ to be causal stable, this pole z_p should be located inside the unit circle, that is

$$|z_p| = \left| \frac{K-2N}{K+2} \right| < 1. \tag{38}$$

Therefore, we should choose

$$K > N - 1. \tag{39}$$

Note, that when $K \to \infty$, then the pole $z_p \to 1$. Since K is an odd number, the minimal K is $K_{\min} = N$ if N is odd and $K_{\min} = N + 1$ if N is even. However, it is more complicated to determine the range of K for the causal stable IIR HB filters in theory when $M \geq 2$. The coefficients of the MF IIR HB filters have been given in (33), then we can investigate what K ensure the IIR HB filters to be causal stable. The obtained result is given in Table I for $1 \leq N \leq 15$ and $1 \leq M \leq 15$. In Table I, the single number in the entry denotes the minimal group delay $K_{\min}$, that is, when $K \geq K_{\min}$, the IIR HB filters are causal stable. The bracketed numbers indicate the range of K and x means that there is no causal stable filter. For example, in the case of $N = 6$ and $M = 4$, the IIR HB filters are causal stable when $K \geq 9$, i.e., $K_{\min} = 9$. In the case of $N = 2$ and $M = 8$, the IIR HB filters become causal stable when $5 \leq K \leq 9$. In the case of $N = 2$ and $M = 12$, there is not any causal stable IIR HB filter. From Table I, we summarize as follows.

1) When $M = 1$, $K_{\min} \geq N$, that is, $K_{\min} = N$ if N is odd and $K_{\min} = N + 1$ if N is even.
2) When $M < N$, $N \leq K_{\min} \leq 2N-1$ and $K_{\min}$ increases as N and M increase.
3) When $M = N$, $K_{\min} = 2N - 1$, because $G(z)$ is an all-pass filter.
4) When $M > N$ but $\not\gg N$, $2N - 1 \leq K_{\min} \leq 2N + 1$. Note that $2N + 1$ is the upper limit of $K_{\min}$.
5) When $M \gg N$, there is no causal stable filter.
6) Between the cases of $M > N$ but $\not\gg N$ and $M \gg N$, there are a few exceptions where the IIR HB filters only with one or some K are causal stable and the lower limit of K is $2N+1$. For example, the filter of $N = 2$, $M = 9$ is causal stable only when $K = 5$.

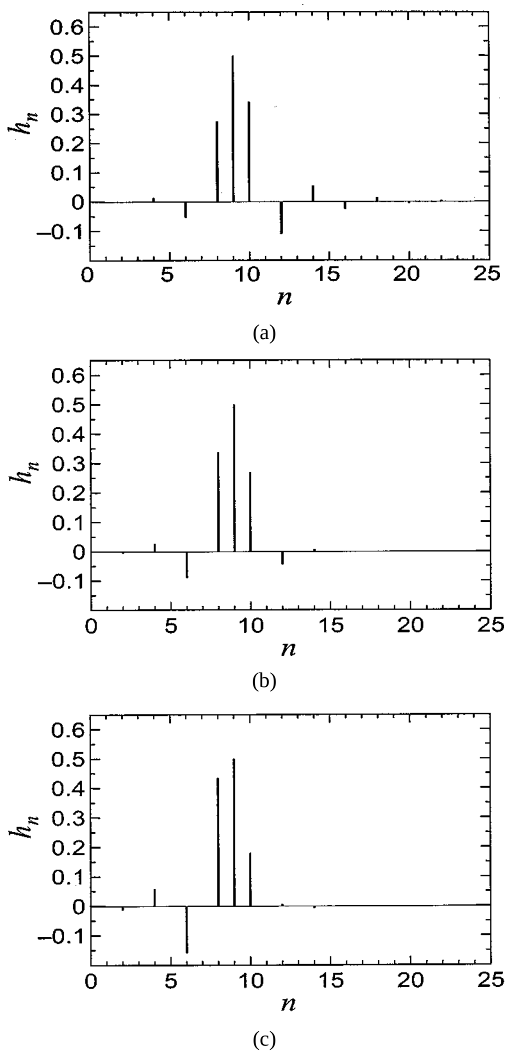

Fig. 2. Impulse responses of MF HB filters in Example 1. (a) $N = 6, M = 2$, (b) $N = 8$ (FIR), (c) $N = M = 4$ (All-pass-based).

E. Exactly Linear Phase IIR HB Filters

When N is odd and M is even, we let $K = N - M$, then the filter coefficients a_n and b_m become (40) seen at the bottom of the page, which satisfy the linear phase conditions of $a_n = a_{N-n}$ and $b_m = b_{M-m}$. Therefore, the MF IIR HB filters have an exactly linear phase response. The exactly linear phase IIR HB filters are not causal since half the poles are located outside the unit circle. However, the noncausal filters can be divided into the causal and anticausal stable parts that have the poles inside and outside the unit circle, respectively, and can be realized in some applications such as image processing and offline processing. The exactly linear phase IIR HB filters are generally needed in image processing applications and have a better magnitude response than the FIR counterparts (see Example 3).

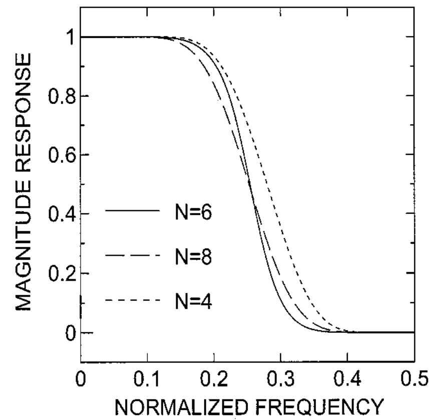

Fig. 3. Magnitude responses of MF HB filters in Example 1.

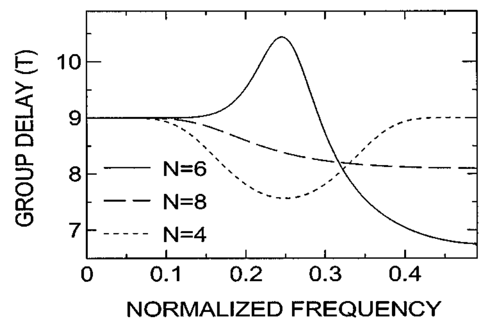

Fig. 4. Group delays of MF HB filters in Example 1.

If $M = 0$, they will degenerate into the conventional FIR HB filters with exactly linear phase. In [13], the MF IIR HB filters with exactly linear phase are also given as a special case of the generalized Butterworth filters and are equivalent to that proposed in this paper with $N > M$. It should be noted that there is no restriction about N and M in (40) and, thus, we can obtain the MF IIR HB filters with $N > M$ and $N < M$. Therefore, the MF IIR HB filters with exactly linear phase proposed in [13] are only a subclass of that given in (40). In addition, if $N = M \pm 1$, all the zeros of $H(z)$ are located on $z = -1$, then the filters are equivalent to the classic Butterworth filters. However, if $N < M - 1$, at least one zero of $H(z)$ is not located on $z = -1$, then they are different from the classic Butterworth filters. Therefore, the exactly linear phase MF IIR HB filters with $N < M - 1$ are new.

VI. Design Examples

Many design examples of the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase and the all-pass-based IIR HB filters have been given in [1]–[10], [12], [15]–[21]. In this section,

$$\begin{cases} a_n = \frac{(-1)^{N-n}}{2} \frac{M!}{n!(N-n)!} \frac{\prod_{i=0}^{N}\left(\frac{N}{2}-i-\frac{M}{2}\right)}{\prod_{i=0}^{M}\left(i-\frac{M}{2}+\frac{N}{2}-n\right)} = \frac{(-1)^{n}}{2} \frac{M!}{n!(N-n)!} \frac{\prod_{i=0}^{N}\left(\frac{N}{2}-i-\frac{M}{2}\right)}{\prod_{i=0}^{M}\left(i-\frac{M}{2}+n-\frac{N}{2}\right)} \\ b_m = (-1)^m \binom{M}{m} \prod_{i=0}^{N} \frac{\frac{N}{2}-i-\frac{M}{2}}{\frac{N}{2}-i+m-\frac{M}{2}} = (-1)^{M-m} \binom{M}{m} \prod_{i=0}^{N} \frac{\frac{N}{2}-i-\frac{M}{2}}{\frac{N}{2}-i+\frac{M}{2}-m} \end{cases} \tag{40}$$

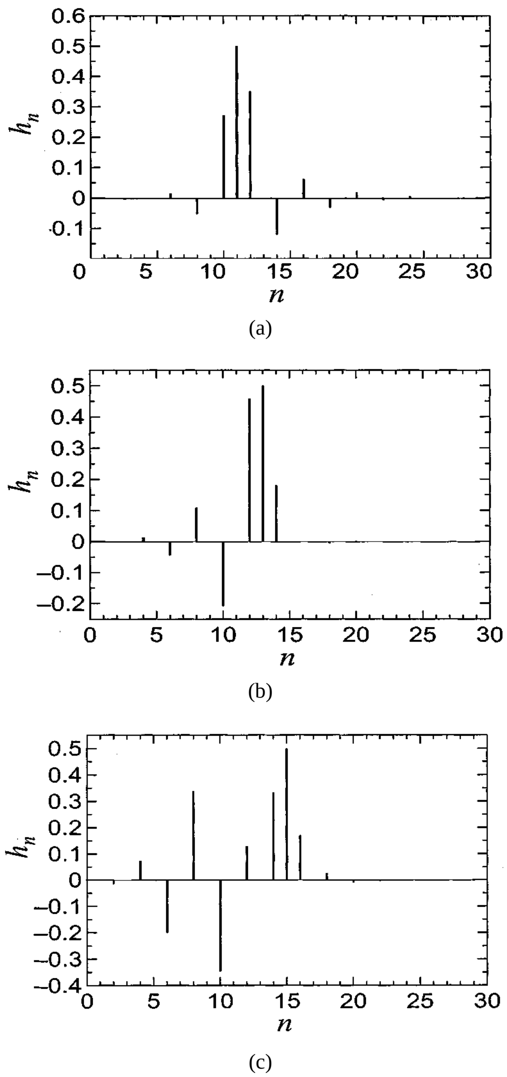

Fig. 5. Impulse responses of MF IIR HB filters in Example 2. (a) $K = 11$, (b) $K = 13$, (c) $K = 15$.

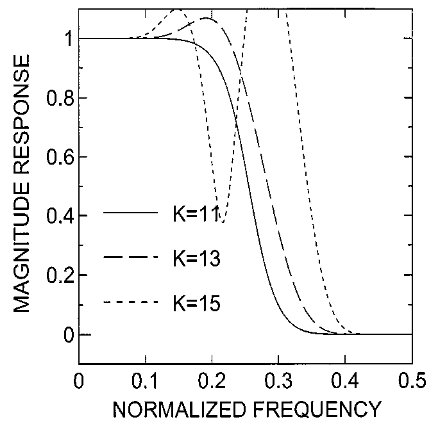

Fig. 6. Magnitude responses of MF IIR HB filters in Example 2.

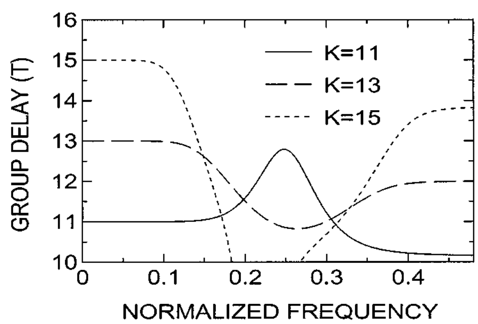

Fig. 7. Group delays of MF IIR HB filters in Example 2.

we present several numerical examples to demonstrate the effectiveness of the proposed MF IIR HB filters and compare the filter performance with the existing MF HB filters.

Example 1: We consider the MF IIR HB filter with $N = 6$ and $M = 2$. The group delay is set to $K = 9$. The filter coefficients can be easily obtained from (33). Its impulse response is shown in Fig. 2(a) and it is clear that it is causal stable. The resulting magnitude and group delay responses are shown in the solid line in Figs. 3 and 4, respectively. For comparison, we have also designed the generalized FIR HB filter of $N = 8$ and the all-pass-based IIR HB filter of $N = M = 4$ with the same group delay of $K = 9$. It is seen from Table I that the all-pass-based IIR HB filter of $N = M = 4$ is causal stable because of $K_{\min} = 7$. These filters have the same flatness since $N + M = 8$ is satisfied. Their impulse responses are shown in Fig. 2 and the magnitude and group delay responses are shown in Figs. 3 and 4 also. It is clear in Figs. 3 and 4 that the proposed IIR HB filter has more flat magnitude response in the passband and stopband and more flat group delay in the passband than the FIR HB filter and the all-pass-based IIR HB filter.

Example 2: We consider the MF IIR HB filters with $N = 6$ and $M = 5$. Firstly, the group delay is set to $K = 11$. It is seen from Table I that the minimal group delay for this filter is $K_{\min} = 11$, then it is causal stable. Its impulse response is shown in Fig. 5(a) and the magnitude and group delay responses are shown in the solid line in Figs. 6 and 7, respectively. We then increase the group delay to $K = 13$. The resulting impulse response is shown in Fig. 5(b) and the magnitude response and group delay are shown in Figs. 6 and 7 also. It is seen in Figs. 6 and 7 that the magnitude response and group delay of the MF IIR HB filter with $K = 11$ are more flat than that with $K = 13$. We have also designed the MF IIR HB filter with $K = 15$. Its impulse response, magnitude response and group delay are shown in Figs. 5(c), 6, and 7, respectively. It is seen that the performance of this filter is very poor. Therefore, we conclude that a larger group delay K will result in a poor frequency response.

Example 3: We consider the MF IIR HB filter with $N = 5$ and $M = 10$. The group delay is set to $K = -5$. Since $K = N - M$ is satisfied, the IIR HB filter has an exactly linear phase response. Its impulse response is shown in Fig. 8(a) and is symmetrical. The resulting magnitude response is shown in the solid line in Fig. 9. We have also designed another IIR HB filter with $N = 11$ and $M = 4$. To get an exactly linear phase response, the group delay is set to $K = N - M = 7$, which is the same as that proposed in [13]. Its impulse response and magnitude response are shown in Figs. 8(b) and 9, respectively. For comparison, the exactly linear phase MF FIR HB filter with $N = K = 15$ has been designed also. The impulse response

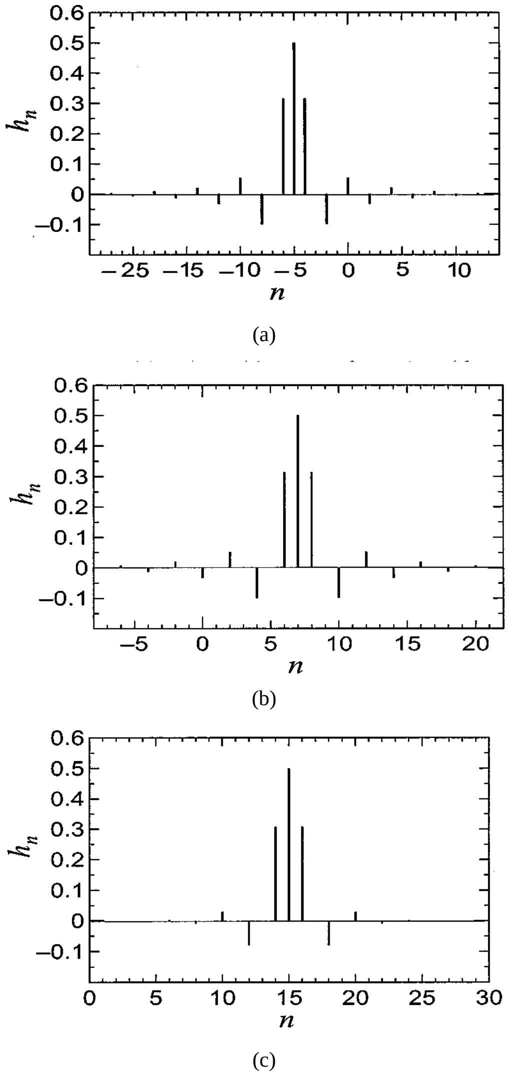

Fig. 8. Impulse responses of the exactly linear phase MF HB filters in Example 3. (a) $N = 5$, $M = 10$, (b) $N = 11$, $M = 4$, (c) $N = 15$ (FIR).

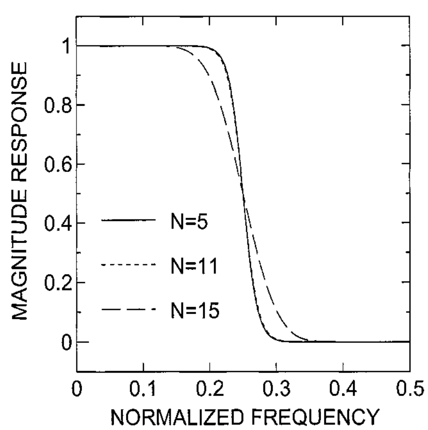

Fig. 9. Magnitude responses of the exactly linear phase MF HB filters in Example 3.

and magnitude response are shown in Figs. 8(c) and 9. Note that these MF HB filters have the same flatness since $N + M = 15$ is satisfied. It is seen in Fig. 9 that the IIR HB filters with exactly linear phase have a better magnitude response than the conventional FIR HB filters with exactly linear phase.

VII. Conclusion

In this paper, we have proposed a more general class of IIR HB filters than the existing HB filters. The proposed IIR HB filters include not only the conventional FIR HB filters with exactly linear phase, the generalized FIR HB filters with approximately linear phase and the all-pass-based IIR HB filters as special cases, but also the causal stable IIR HB filters and the IIR HB filters with exactly linear phase. We have given a new closed-form expression for the transfer function of the MF IIR HB filters. The filter coefficients are directly obtained by solving a linear system of Vandermonde equations from the maximal flatness conditions. We have also investigated the conditions for realizing the causal stable IIR HB filters and the IIR HB filters with exactly linear phase. It has been shown through the design examples that the causal stable MF IIR HB filters can outdo the FIR HB filters and the all-pass-based IIR HB filters with the same flatness in general, while the MF IIR HB filters with exactly linear phase have a better magnitude response than the FIR counterparts. It has been found also that the frequency response of IIR HB filters becomes very poor as the group delay increases.

References

[1] A. Antoniou, *Digital Filters: Analysis, Design and Applications*. New York: McGraw-Hill, 1993.

[2] S. K. Mitra and J. F. Kaiser, *Handbook for Digital Signal Processing*. New York: Wiley, 1993.

[3] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[4] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Cambridge, MA: Wellesley-Cambridge, 1996.

[5] B. A. Shenoi, *Magnitude and Delay Approximation of 1-D and 2-D Digital Filters*. Berlin, Germany: Springer-Verlag, 1999.

[6] O. Herrmann, "On the approximation problem in nonrecursive digital filter design," *IEEE Trans. Circuits Theory*, vol. 18, pp. 411–413, May 1971.

[7] F. Mintzer, "On half-band, third-band and Nth band FIR filters and their design," *IEEE Trans. Acoust. Speech Signal Processing*, vol. ASSP-30, pp. 734–738, Oct. 1982.

[8] P. P. Vaidyanathan and T. Q. Nguyen, "A "trick" for the design of FIR half-band filters," *IEEE Trans. Circuits Syst. II*, vol. CAS-34, pp. 297–300, Mar. 1987.

[9] M. Vetterli and C. Herley, "Wavelets and filter banks: Theory and design," *IEEE Trans. Signal Processing*, vol. 40, pp. 2207–2232, Sept. 1992.

[10] S. M. Phoong, C. W. Kim, P. P. Vaidyanathan, and R. Ansari, "A new class of two channel biorthogonal filter banks and wavelet bases," *IEEE Trans. Signal Processing*, vol. 43, pp. 649–665, Mar. 1995.

[11] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol. 13, pp. 30–60, Jan. 1996.

[12] I. W. Selesnick, "Formulas for orthogonal IIR wavelet filters," *IEEE Trans. Signal Processing*, vol. 46, pp. 1138–1141, Apr. 1998.

[13] I. W. Selesnick and C. S. Burrus, "Generalized digital Butterworth filter design," *IEEE Trans. Signal Processing*, vol. 46, pp. 1688–1694, June 1998.

[14] X. Zhang and H. Iwakura, "Design of IIR digital all-pass filters based on eigenvalue problem," *IEEE Trans. Signal Processing*, vol. 47, pp. 554–559, Feb. 1999.

[15] S. Samadi, A. Nishihara, and H. Iwakura, "Generalized half-band maximally flat FIR filters," in *Proc. ISCAS*, vol. III, May 1999, pp. 279–282.

[16] X. Zhang and T. Yoshikawa, "Design of FIR Nyquist filters with low group delay," *IEEE Trans. Signal Processing*, vol. 47, pp. 1454–1458, May 1999.

[17] Y. Wisutmethangoon and T. Q. Nguyen, "A method for design of M th band filters," *IEEE Trans. Signal Processing*, vol. 47, pp. 1669–1678, June 1999.

[18] S. Samadi, H. Iwakura, and A. Nishihara, "Multiplierless and hierarchical structures for maximally flat half-band FIR filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 1225–1230, Sept. 1999.

[19] X. Zhang, K. Inotsume, and T. Yoshikawa, "Design of low delay FIR half-band filters with arbitrary flatness and its application to filter banks," *IEICE Trans. Commun.*, vol. J82-A, no. 10, pp. 1529–1537, Oct. 1999.

[20] S. C. Pei and P. H. Wang, "Closed-form design and efficient implementation of generalized maximally flat half-band FIR filters," *IEEE Signal Processing Lett.*, vol. 7, pp. 149–151, June 2000.

[21] S. Samadi, A. Nishihara, and H. Iwakura, "Universal maximally flat lowpass FIR systems," *IEEE Trans. Signal Processing*, vol. 48, pp. 1956–1963, July 2000.

[22] H. Schuessler and P. Steffan, "Recursive halfband filters," *Int. J. Electron. Commun. (AEU)*, vol. 55, no. 6, pp. 1–12, 2001.

[23] A. D. Poularikas, *The Handbook of Formulas and Tables for Signal Processing*. Boca Raton, FL: CRC, 1999.

Xi Zhang (M'94–SM'01) received the B.E. degree in electronics engineering from Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 1984 and the M.E. and Ph.D. degrees in communication engineering from the University of Electro-Communications (UEC), Tokyo, Japan, in 1990 and 1993, respectively.

From 1984 to 1987, he was an Assistant Professor with the Department of Electronics Engineering at NUAA. From 1993 to 1996, he was an Assistant Professor with the Department of Communications and Systems at UEC. From 2000 to 2001, he was a Visiting Scientist at the Massachusetts Institute of Technology, Cambridge, as a representative of the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. Currently, he is an Associate Professor with the Department of Electrical Engineering, Nagaoka University of Technology, Niigata, Japan. His research interests are in the areas of digital signal processing, filter design theory, filter banks and wavelets and its applications to image coding. Since 2002, he has been an Associate Editor for the IEEE SIGNAL PROCESSING LETTERS.

Dr. Zhang is a Member of the Institute of Electronics and Communications Engineers of Japan. He was awarded Third Prize from the Science and Technology Progress Award of China in 1987 and the Challenge Prize of 4th LSI IP Design Award of Japan in 2002.

Kevin Amaratunga received the Ph.D. degree from the Massachusetts Institute of Technology, Cambridge.

Currently, he is an Assistant Professor of information technology in the Department of Civil and Environmental Engineering at the Massachusetts Institute of Technology. His research focuses on wavelets and filter banks and their applications to engineering information systems (especially digital imaging) and high-performance computer simulation.