A New Class of Orthonormal Symmetric Wavelet Bases Using a Complex Allpass Filter

Xi Zhang, *Senior Member, IEEE*, Akira Kato, and Toshinori Yoshikawa, *Member, IEEE*

***Abstract*—This paper considers the design of the WSS paraunitary filterbanks composed of a single complex allpass filter and gives a new class of real-valued orthonormal symmetric wavelet bases. First, the conditions that the complex allpass filter has to satisfy are derived from the symmetry and orthonormality conditions of wavelets, and its transfer function is given to satisfy these conditions. Second, the paraunitary filter banks are designed by using the derived transfer function from the viewpoints of the regularity and frequency selectivity. A new method for designing the proposed paraunitary filterbanks with a given degrees of flatness is presented. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm. Therefore, the filter coefficients can be easily obtained by solving the eigenvalue problem, and the optimal solution is attained through a few iterations. Furthermore, both the maximally flat and minimax solutions are also included in the proposed method as two specific cases. The maximally flat filters have a closed-form solution without any iteration. Finally, some design examples are presented to demonstrate the effectiveness of the proposed method.**

***Index Terms*—Complex allpass filter, eigenvalue problem, orthonormal symmetric wavelets, Remez exchange algorithm.**

I. Introduction

The discrete wavelet transform (DWT) has been applied extensively to digital signal and image processing [1]–[20]. In many applications, wavelets are required to be real since the signals to be processed are real-valued in general. In this paper, we restrict ourselves to real-valued wavelets. It is well-known [1], [7], [11] that the real-valued orthonormal wavelet bases can be generated by two-band paraunitary filterbanks with real coefficients. One desirable property for wavelets is symmetry, which requires all filters in the filterbanks to possess exactly linear phase because the symmetric extension method is generally used to treat the boundaries of images in digital image coding [9], [10], [16]. It is known [2] that FIR filters (corresponding to the compactly supported wavelets) can easily realize the linear phase. However, it is widely appreciated [1], [4] that the only FIR solution that produces a real-valued orthonormal symmetric wavelet basis is the Haar solution, which is not continuous. To obtain real-valued orthonormal symmetric wavelet bases with more regularity than the Haar solution, Herley and Vetterli have proposed a class of IIR solutions in [12]. In [12], Herley and Vetterli discussed two cases: half sample symmetric (HSS) and whole sample symmetric (WSS). In the HSS case, the scaling and wavelet functions are symmetric and antisymmetric respectively, whereas both are symmetric in the WSS case. Herley and Vetterli have shown in [12] that the HSS filterbanks can be constructed by using real allpass filters. The design methods for these allpass-based HSS filterbanks have been proposed in [15] and [20]. However, the WSS filterbanks are not as easy to use as the HSS ones, and Herley and Vetterli gave only one example.

In this paper, we discuss the WSS case and give a new class of real-valued orthonormal symmetric wavelet bases, where the associated paraunitary filterbanks are composed of a single complex allpass filter [6], [19]. First, we derive the conditions imposed on the complex allpass filter from the symmetry and orthonormality conditions of wavelets and give the transfer function of complex allpass filter to satisfy these conditions. Second, we consider the design of the paraunitary filterbanks by using the derived transfer function from the viewpoints of the regularity and frequency selectivity [13], [20]. We propose a new method for designing the paraunitary filterbanks with a given degrees of flatness. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm [14], [18]. Therefore, the filter coefficients can be easily obtained by solving the eigenvalue problem to find the positive minimum eigenvalue, and the optimal solution is attained through a few iterations. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step. Furthermore, both the maximally flat and minimax solutions are also included in the proposed method as two specific cases. The maximally flat filters have a closed-form solution without any iteration. Finally, some design examples are presented to demonstrate the effectiveness of the proposed method. The major contributions in this paper are the closed-form solution for the maximally flat wavelet filters and the design method using the Remez exchange algorithm for the wavelet filters with a given degrees of flatness.

This paper is organized as follows. In Section II, the conditions imposed on the complex allpass filter are derived from the symmetry and orthonormality conditions of wavelets, and its transfer function is given to satisfy these conditions. A closed-form solution for the maximally flat filters is presented in Section III. In Section IV, a new method for designing the paraunitary filterbanks with a given degree of flatness is proposed, based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm. Some numerical examples are presented in Section V.

Manuscript received December 8, 1999; revised August 2, 2001. The associate editor coordinating the review of this paper and approving it for publication was Dr. Xiang-Gen Xia.

The authors are with the Department of Electrical Engineering, Nagaoka University of Technology, Nagaoka, Niigata, Japan (e-mail: xiz@nagaokaut.ac.jp; kakira@inflab21.nagaokaut.ac.jp; tyoshi@nagaokaut.ac.jp).

Publisher Item Identifier S 1053-587X(01)09225-X.

1053–587X/01$10.00 © 2001 IEEE

II. Orthonormal Symmetric Wavelets

It is well-known [1], [4], [7] that a real-valued orthonormal wavelet basis can be generated by a two-band paraunitary filterbank $\{H(z), G(z)\}$ with real coefficients, where $H(z)$ is assumed to be a lowpass filter, and $G(z)$ is highpass. The orthonormality condition that $H(z)$ and $G(z)$ have to satisfy is

$$\begin{cases} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1 \\ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 1 \\ H(z)G(z^{-1}) + H(-z)G(-z^{-1}) = 0. \end{cases} \tag{1}$$

In image coding applications, the circular extension and symmetric extension methods are generally used to treat the boundaries of images. The symmetric extension has been proven to yield a better quality of the reconstructed image [9], [10]. The symmetric extension method requires the wavelet basis to be symmetric, i.e., both $H(z)$ and $G(z)$ possess exactly linear phase response. To satisfy the symmetry and orthonormality conditions simultaneously, Herley and Vetterli have proposed a class of linear-phase IIR solutions in [12]. In [12], the half sample symmetric (HSS) and whole sample symmetric (WSS) cases are discussed. It is shown that the HSS filterbanks can be constructed by using real allpass filters, where the numerator degree of $H(z)$ and $G(z)$ is odd, resulting in a symmetric scaling function and an antisymmetric wavelet function. However, the WSS filterbanks are not as easy to use as the HSS ones, and only one example is given in [12]. In this paper, we will consider the WSS filterbanks, i.e., the numerator degree of $H(z)$ and $G(z)$ is even, and both the scaling and wavelet functions are symmetric.

According to [6], we construct $H(z)$ and $G(z)$ by using a single complex allpass filter as follows:

$$\begin{cases} H(z) = \frac{1}{2}\{A(z) + \tilde{A}(z)\} \\ G(z) = \frac{z^{-1}}{2j}\{A(z) - \tilde{A}(z)\} \end{cases} \tag{2}$$

where $A(z)$ is a complex allpass filter, and $\tilde{A}(z)$ has a set of filter coefficients that are complex conjugate with ones of $A(z)$. One can verify that $H(z)$ and $G(z)$ have a set of real-valued filter coefficients and the numerator degree is even. From the orthonormality condition in (1), $A(z)$ must satisfy [6]

$$A(z) = \pm j\tilde{A}(-z) \tag{3}$$

which means that if α is a pole of $A(z)$, then $-\alpha^*$ is also a pole of $A(z)$. Consequently, $A(z)$ has a pair of poles $(\alpha, -\alpha^*)$ and/or one pole $j\beta$, where α is complex, β is real, and α^* denotes the complex conjugate of α. To force $H(z)$ and $G(z)$ to have exactly linear phase response, $A(z)$ must also satisfy

$$A(z) = \frac{1}{\tilde{A}(z)}. \tag{4}$$

It is known that $A(z)$ and $\tilde{A}(z)$ satisfy the following relation:

$$A(z) = \frac{1}{\tilde{A}(z^{-1})}. \tag{5}$$

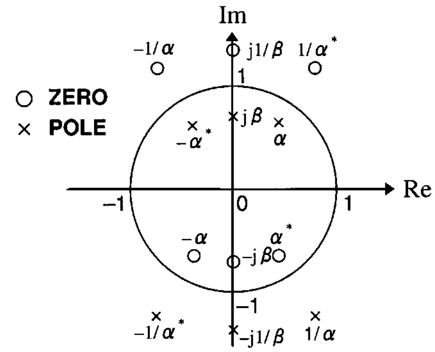

Fig. 1. Pole-zero location of $A(z)$.

Thus, the condition of (4) becomes

$$A(z) = A(z^{-1}) \tag{6}$$

which means that if α is a pole of $A(z)$, then $1/\alpha$ is also a pole of $A(z)$. Therefore, $A(z)$ has a quadruplet of poles $(\alpha, 1/\alpha, -\alpha^*, -1/\alpha^*)$ and/or a pair of poles $(j\beta, 1/j\beta)$, as shown in Fig. 1. The transfer function of $A(z)$ can be expressed as

$$\begin{aligned} A(z) = e^{j\eta}z^{-N} \prod_{k=1}^{N_1} \frac{(1 + j\beta_k z)\left(1 - j\beta_k^{-1} z\right)}{(1 - j\beta_k z^{-1})\left(1 + j\beta_k^{-1} z^{-1}\right)} \\ \times \prod_{k=1}^{N_2} \frac{(1 - \alpha_k^* z)\left(1 - \frac{z}{\alpha_k^*}\right)(1 + \alpha_k z)\left(1 + \frac{z}{\alpha_k}\right)}{(1 - \alpha_k z^{-1})\left(1 - \frac{z^{-1}}{\alpha_k}\right)(1 + \alpha_k^* z^{-1})\left(1 + \frac{z^{-1}}{\alpha_k^*}\right)} \end{aligned} \tag{7}$$

where the degree of $A(z)$ is $N = 2N_1 + 4N_2$, and $\eta = \pm\pi/4$ or $\pm 3\pi/4$. By expanding (7), we have

$$\begin{aligned} A(z) = e^{j\eta}z^{-N} \frac{a_0 + ja_1 z + a_2 z^2 + \cdots}{a_0 - ja_1 z^{-1} + a_2 z^{-2} + \cdots} \\ \times \frac{\cdots + a_2 z^{N-2} + ja_1 z^{N-1} + a_0 z^N}{\cdots + a_2 z^{-N+2} - ja_1 z^{-N+1} + a_0 z^{-N}} \end{aligned} \tag{8}$$

where a_n are real coefficients, and $a_0 = 1$. It should be noted that the symmetry and orthonormality conditions have been satisfied by using the transfer function of $A(z)$ given in (8). Therefore, the design problem of the paraunitary filterbank $\{H(z), G(z)\}$ with exactly linear phase becomes the phase approximation of $A(z)$ in (8).

Let $\theta(\omega)$ be the phase response of $A(z)$. We have from (8)

$$\theta(\omega) = \eta + 2\varphi(\omega) \tag{9}$$

where when $M = N/2$ is even

$$\begin{aligned} \varphi(\omega) &= \tan^{-1} \frac{2\sum_{n=0}^{M/2-1} a_{2n+1}\cos(M - 2n - 1)\omega}{a_M + 2\sum_{n=0}^{M/2-1} a_{2n}\cos(M - 2n)\omega} \\ &= \tan^{-1}\frac{N(\omega)}{D(\omega)} \end{aligned} \tag{10}$$

and when $M = N/2$ is odd

$$\varphi(\omega) = \tan^{-1} \frac{a_M + 2\sum_{n=0}^{(M-3)/2} a_{2n+1}\cos(M-2n-1)\omega}{2\sum_{n=0}^{(M-1)/2} a_{2n}\cos(M-2n)\omega} = \tan^{-1}\frac{N(\omega)}{D(\omega)}. \quad (11)$$

Therefore, we have from (2)

$$\begin{cases} H(e^{j\omega}) = \cos\theta(\omega) \\ G(e^{j\omega}) = e^{-j\omega}\sin\theta(\omega). \end{cases} \quad (12)$$

It is clear that both $H(z)$ and $G(z)$ have exactly linear-phase response, and their magnitude responses satisfy the following power-complementary relation:

$$|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1. \quad (13)$$

III. Maximally Flat Filters

In this section, we describe the design of the proposed WSS filterbanks with the maximum flatness. From the viewpoint of the regularity, $H(z)$ and $G(z)$ are required to meet the following flatness conditions:

$$\left.\frac{\partial^k |H(e^{j\omega})|}{\partial\omega^k}\right|_{\omega=\pi} = 0 \quad (k = 0, 1, \ldots, K-1) \quad (14)$$

$$\left.\frac{\partial^k |G(e^{j\omega})|}{\partial\omega^k}\right|_{\omega=0} = 0 \quad (k = 0, 1, \ldots, K-1) \quad (15)$$

where K is even, and $0 \le K \le N$. Note that $K = N$ corresponds to the maximally flat filters, and when $K = 0$, there is no flatness condition imposed on $H(z)$ and $G(z)$. Equations (14) and (15) imply that $H(z)$ and $G(z)$ contain K zeros located at $z = -1$ and $z = 1$, respectively. Note that since $H(z)$ and $G(z)$ are orthogonal, the flatness conditions in (14) and (15) are equivalent to each other. For convenience, we use the condition in (15). Substituting directly the magnitude response of $G(z)$ into (15) will result in a set of nonlinear equations to be solved, which is very difficult when N is large. To avoid this problem, we decompose $|G(e^{j\omega})|$ as

$$|G(e^{j\omega})| = \sin\theta(\omega) = 2\sin\frac{\theta(\omega)}{2}\cos\frac{\theta(\omega)}{2} = 2|G_1(e^{j\omega})||G_2(e^{j\omega})| \quad (16)$$

where

$$|G_1(e^{j\omega})| = \sin\frac{\eta}{2}\cos\varphi(\omega) + \cos\frac{\eta}{2}\sin\varphi(\omega) = \frac{\sin\frac{\eta}{2}D(\omega) + \cos\frac{\eta}{2}N(\omega)}{\sqrt{N(\omega)^2 + D(\omega)^2}} \quad (17)$$

$$|G_2(e^{j\omega})| = \cos\frac{\eta}{2}\cos\varphi(\omega) - \sin\frac{\eta}{2}\sin\varphi(\omega) = \frac{\cos\frac{\eta}{2}D(\omega) - \sin\frac{\eta}{2}N(\omega)}{\sqrt{N(\omega)^2 + D(\omega)^2}}. \quad (18)$$

By differentiating (16), we get

$$\frac{\partial^k |G(e^{j\omega})|}{\partial\omega^k} = 2\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^i |G_1(e^{j\omega})|}{\partial\omega^i}\frac{\partial^{k-i}|G_2(e^{j\omega})|}{\partial\omega^{k-i}}. \quad (19)$$

Since $G(z)$ is a highpass filter, it is clear from (12) that since $\theta(0) = 0$ is required, then $|G_2(1)| = 1$. Therefore, it can be easily derived from (19) that the flatness condition of (15) is equivalent to

$$\left.\frac{\partial^k |G_1(e^{j\omega})|}{\partial\omega^k}\right|_{\omega=0} = 0 \quad (k = 0, 1, \ldots, K-1). \quad (20)$$

Similarly, it is seen from (17) that the condition of (20) can be reduced to

$$\left.\frac{\partial^k\left[\sin\frac{\eta}{2}D(\omega) + \cos\frac{\eta}{2}N(\omega)\right]}{\partial\omega^k}\right|_{\omega=0} = 0 \quad (k = 0, 1, \ldots, K-1) \quad (21)$$

that is

$$\left.\frac{\partial^k D(\omega)}{\partial\omega^k}\right|_{\omega=0} + \cot\frac{\eta}{2}\left.\frac{\partial^k N(\omega)}{\partial\omega^k}\right|_{\omega=0} = 0 \quad (k = 0, 1, \ldots, K-1) \quad (22)$$

and $D(\omega)$ in (10) or (11) into (22), we can rewrite (22) in matrix form as

$$\boldsymbol{V}D a = \boldsymbol{0} \quad (23)$$

where $\boldsymbol{a} = [a_0, a_1, \ldots, a_M]^T, \boldsymbol{0} = [0, 0, \ldots, 0]^T$

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ M^2 & (M-1)^2 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M^{K-2} & (M-1)^{K-2} & \cdots & 1 & 0 \end{bmatrix} \quad (24)$$

and $\boldsymbol{D} = \mathrm{diag}[d_0, d_1, \ldots, d_M]$

$$d_i = \begin{cases} 1, & (i: \text{even}) \\ \cot\frac{\eta}{2}, & (i: \text{odd}) \end{cases} \quad (25)$$

for $i = 0, 1, \ldots, M-1$, and

$$d_M = \begin{cases} \frac{1}{2}, & (M: \text{even}) \\ \frac{1}{2}\cot\frac{\eta}{2}, & (M: \text{odd}). \end{cases} \quad (26)$$

When the maximally flat filters, i.e., $K = N = 2M$ are required, there is always a unique solution in (23) due to $a_0 = 1$. Therefore, the maximally flat solutions can be obtained by solving the above linear equations. However, it should be noticed that $\boldsymbol{V}$ is a Vandermonde matrix and can be analytically solved. Thus, the closed-form formula is given by

$$a_n = \begin{cases} \binom{N}{n}, & (n: \text{even}) \\ -\binom{N}{n}\tan\frac{\eta}{2}, & (n: \text{odd}) \end{cases} \quad (27)$$

for $i = 0, 1, \ldots, N$.

IV. Filter Design with Given Flatness

It is well-known that the maximally flat filters have poor frequency selectivity. Of course, frequency selectivity is also thought of as a useful property for many applications. However, it is known in [13] that regularity and frequency selectivity somewhat contradict each other. For this reason, we consider the design of the paraunitary filterbanks that have the best possible frequency selectivity for a given regularity, i.e., a given degrees of flatness. The flatness conditions have been given in (14) and (15), but $0 \leq K < N$ in this case. Our aim is to achieve an equiripple response by using the remaining degrees of freedom. $H(z)$ and $G(z)$ are required to be a pair of lowpass and highpass filters. The desired magnitude responses are given by

$$|H_d(e^{j\omega})| = \begin{cases} 1 & (0 \leq \omega \leq \omega_p) \\ 0 & (\omega_s \leq \omega \leq \pi) \end{cases} \tag{28}$$

$$|G_d(e^{j\omega})| = \begin{cases} 0 & (0 \leq \omega \leq \omega_p) \\ 1 & (\omega_s \leq \omega \leq \pi) \end{cases} \tag{29}$$

where ω_p and ω_s are the cut-off frequencies of the passband and stopband of $H(z)$, respectively, and $\omega_p + \omega_s = \pi$. Therefore, the desired phase response of $A(z)$ is from (12)

$$\theta_d(\omega) = \begin{cases} 0 & (0 \leq \omega \leq \omega_p) \\ \pm\dfrac{\pi}{2} & (\omega_s \leq \omega \leq \pi) \end{cases} \tag{30}$$

and the desired response of $\varphi(\omega)$ is from (9)

$$\varphi_d(\omega) = \begin{cases} -\dfrac{\eta}{2} & (0 \leq \omega \leq \omega_p) \\ \pm\dfrac{\pi}{4} - \dfrac{\eta}{2} & (\omega_s \leq \omega \leq \pi). \end{cases} \tag{31}$$

When M is even, it is seen from (10) that $\varphi(\pi/2) = 0$ and $\varphi(\omega) = -\varphi(\pi-\omega)$. Thus, we should choose $\eta = \pm\pi/4$ to meet this symmetry property. When M is odd, $\varphi(\pi/2) = \pm\pi/2$ and $\varphi(\omega) = \pm\pi - \varphi(\pi-\omega)$ from (11), then $\eta = \pm 3\pi/4$. Due to the symmetry of $\varphi(\omega)$, we need to approximate $\varphi(\omega)$ to $\varphi_d(\omega)$ in the passband only. Therefore, the design problem becomes the approximation of $\varphi(\omega)$ in the passband.

We use the Remez exchange algorithm and formulate the design problem in the form of the eigenvalue problem. First, we select $(M - K/2 + 1)$ extremal frequencies ω_i in the passband $[0, \omega_p]$ as follows:

$$\omega_p = \omega_0 > \omega_1 > \cdots > \omega_{M-K/2} \geq 0 \tag{32}$$

where when $K > 0$, we should choose $\omega_{M-K/2} > 0$ due to the flatness condition at $\omega = 0$. When $K = 0$, it can be seen that there is not any flatness condition imposed on $H(z)$ and $G(z)$, that is, $H(z)$ and $G(z)$ will not contain any zero located at $z = -1$ and $z = 1$. Thus, we should choose $\omega_M = 0$, which results in the optimal (minimax) solution in the Chebyshev sense. We then formulate $\varphi(\omega)$ as

$$\varphi(\omega_i) - \varphi_d(\omega_i) = (-1)^{i+l}\delta_p \tag{33}$$

where δ_p is a phase error to be minimized, and $l = 0$ or 1 to guarantee $\delta_p > 0$. According to the symmetry of $\varphi(\omega)$, when M is even, we have $l = 0$ if $\eta = -\pi/4$, and $l = 1$ if $\eta = \pi/4$,

TABLE I
Filter Coefficients of $A(z)$ in Example 1

	$N=2$	$N=4$	$N=6$	$N=8$
	$\eta = -\frac{3}{4}\pi$	$\eta = \frac{1}{4}\pi$	$\eta = -\frac{3}{4}\pi$	$\eta = \frac{1}{4}\pi$
a_0	1.000000	1.000000	1.000000	1.000000
a_1	4.828427	−1.656854	14.485281	−3.313708
a_2	1.000000	6.000000	15.000000	28.000000
a_3		−1.656854	48.284271	−23.195959
a_4		1.000000	15.000000	70.000000
a_5			14.485281	−23.195959
a_6			1.000000	28.000000
a_7				−3.313708
a_8				1.000000

whereas when M is odd, $l = 0$ if $\eta = 3\pi/4$, and $l = 1$ if $\eta = -3\pi/4$. From (10) or (11), we have

$$\begin{aligned} \tan[\varphi(\omega_i) - \varphi_d(\omega_i)] &= \frac{\tan\varphi(\omega_i) - \tan\varphi_d(\omega_i)}{1 + \tan\varphi(\omega_i)\tan\varphi_d(\omega_i)} \\ &= \frac{D(\omega_i) + N(\omega_i)\cot\frac{\eta}{2}}{D(\omega_i)\cot\frac{\eta}{2} - N(\omega_i)} = (-1)^{i+l}\delta \end{aligned} \tag{34}$$

where $\delta = \tan\delta_p$, and the denominator polynomial must satisfy

$$D(\omega)\cot\frac{\eta}{2} - N(\omega) \neq 0 \quad (\forall\omega). \tag{35}$$

Substituting $N(\omega)$Equation (34) can be rewritten in matrix form as

$$\boldsymbol{C}Da = \delta\boldsymbol{C}Ta \tag{36}$$

where

$$\boldsymbol{C} = \begin{bmatrix} \cos M\omega_0 & \cos(M-1)\omega_0 & \cdots & 1 \\ \cos M\omega_1 & \cos(M-1)\omega_1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \cos M\omega_{M-K/2} & \cos(M-1)\omega_{M-K/2} & \cdots & 1 \end{bmatrix} \tag{37}$$

and $\boldsymbol{T} = \mathrm{diag}[t_0, t_1, \ldots, t_M]$

$$t_i = \begin{cases} (-1)^l \cot\dfrac{\eta}{2}, & (i: \text{even}) \\ (-1)^{l+1}, & (i: \text{odd}) \end{cases} \tag{38}$$

for $i = 0, 1, \ldots, M-1$, and

$$t_M = \begin{cases} \dfrac{(-1)^l}{2}\cot\dfrac{\eta}{2}, & (M: \text{even}) \\ \dfrac{(-1)^{l+1}}{2}, & (M: \text{odd}). \end{cases} \tag{39}$$

By involving the flatness condition in (23), we have

$$\boldsymbol{P}a = \delta\boldsymbol{Q}a \tag{40}$$

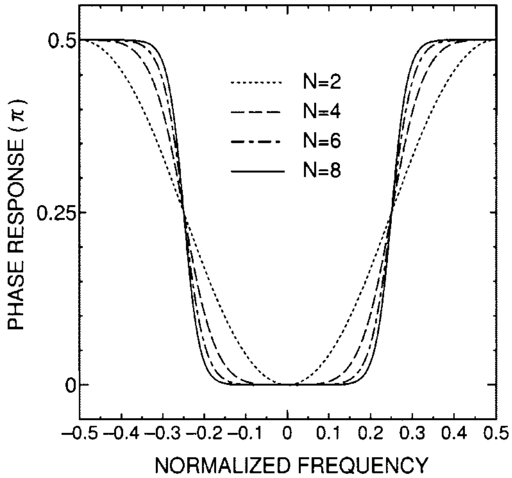

Fig. 2. Phase responses of $A(z)$ in Example 1.

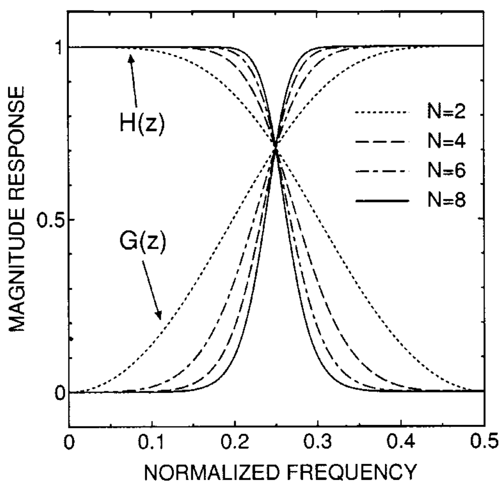

Fig. 3. Magnitude responses of $H(z)$ and $G(z)$ in Example 1.

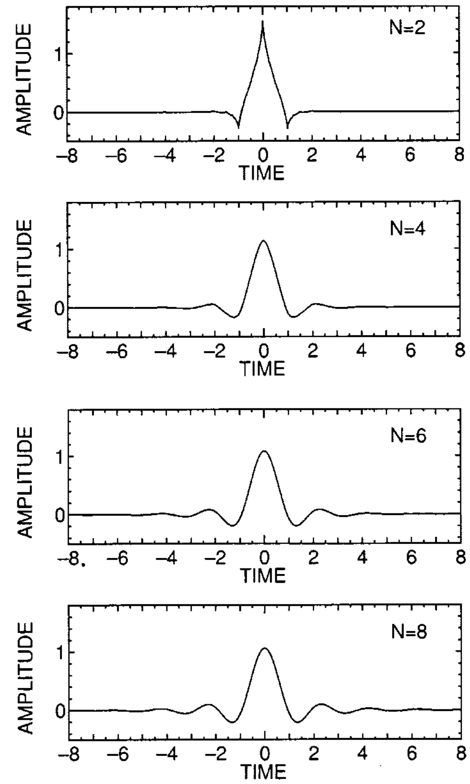

Fig. 4. Scaling functions in Example 1.

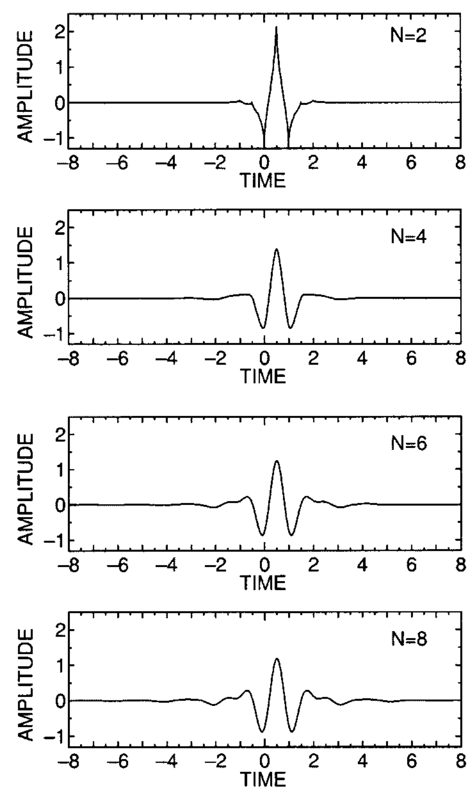

Fig. 5. Wavelet functions in Example 1.

where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{C} \end{bmatrix} \boldsymbol{D} \tag{41}$$

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C} \end{bmatrix} \boldsymbol{T} \tag{42}$$

and $\mathbf{0}$ is a $K/2 \times (M+1)$ null matrix. It should be noted that (40) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\boldsymbol{a}$ is a corresponding eigenvector. In order to minimize δ (>0), we must find the positive minimum eigenvalue by solving the above eigenvalue problem [14], [18], which can be done efficiently by using the iterative power method, so that the corresponding eigenvector gives a set of filter coefficients a_n. By using the obtained filter coefficients a_n, we compute the phase response $\varphi(\omega)$ and search for all extremal frequencies Ω_i in the passband. As a result, it could be found that the obtained $\varphi(\omega)$ may not be equiripple. We then choose the extremal frequencies Ω_i as the sampling frequencies ω_i in the next iteration and solve the eigenvalue problem of (40) to get a set of filter coefficients a_n again. The above procedure is iterated until the equiripple response is attained. The convergence of the proposed algorithm has been proven in [21], provided that (35) is satisfied. It has also been proven in [14] and [18] that (35) can be satisfied by choosing the positive minimum eigenvalue if the solution that satisfies (35) exists. However, sometimes none of the eigenvalues gives a solution that satisfies (35). That is,

TABLE II
FILTER COEFFICIENTS OF $A(z)$ IN EXAMPLE 2

	$K=0$	$K=2$	$K=4$	$K=6$
a_0	1.000000	1.000000	1.000000	1.000000
a_1	6.990896	7.751857	10.633790	14.485281
a_2	5.289258	5.730382	8.618640	15.000000
a_3	15.177506	16.993447	25.175324	48.284271
a_4	5.289258	5.730382	8.618640	15.000000
a_5	6.990896	7.751857	10.633790	14.485281
a_6	1.000000	1.000000	1.000000	1.000000

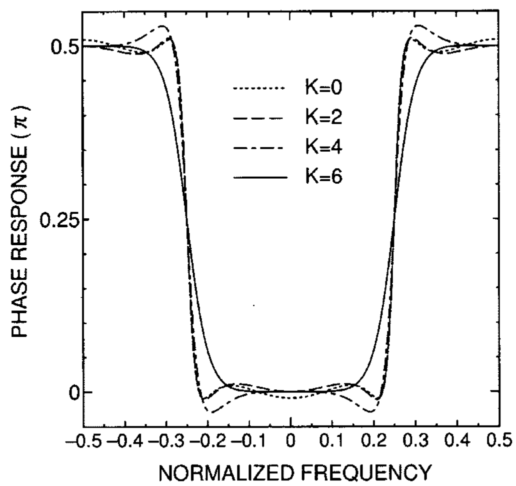

Fig. 6. Phase responses of $A(z)$ in Example 2.

the solution that satisfies (35) may not exist. In this case, the algorithm fails to converge. In general, it is caused by an unapt choice of the extremal frequency points. Therefore, there always exists the solution that satisfies (35) by appropriately choosing the extremal frequencies, as shown in (32). The design algorithm is shown in detail as follows.

Procedure {Design Algorithm for Complex Allpass Filters}

Begin

1) Read N, K and the cutoff frequency ω_p.

2) Select an initial extremal frequencies Ω_i $(i = 0, 1, \ldots, M - K/2)$ equally spaced in the passband.

Repeat

3) Set $\omega_i = \Omega_i$ for $i = 0, 1, \ldots, M - K/2$.

4) Compute $\boldsymbol{P}$ and $\boldsymbol{Q}$ by using (41) and (42), then find the positive minimum eigenvalue of (40) to obtain a set of filter coefficients a_n.

5) Compute the phase response $\varphi(\omega)$ and search the extremal frequencies Ω_i in the passband.

Until Satisfy the following condition for a prescribed small constant ϵ (typically, $\epsilon = 10^{-4}$)

$$\sum_{i=0}^{M-K/2} |\Omega_i - \omega_i| \leq \epsilon$$

End.

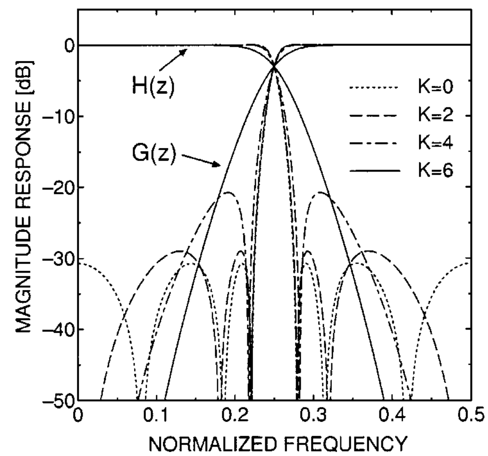

Fig. 7. Magnitude responses of $H(z)$ and $G(z)$ in Example 2.

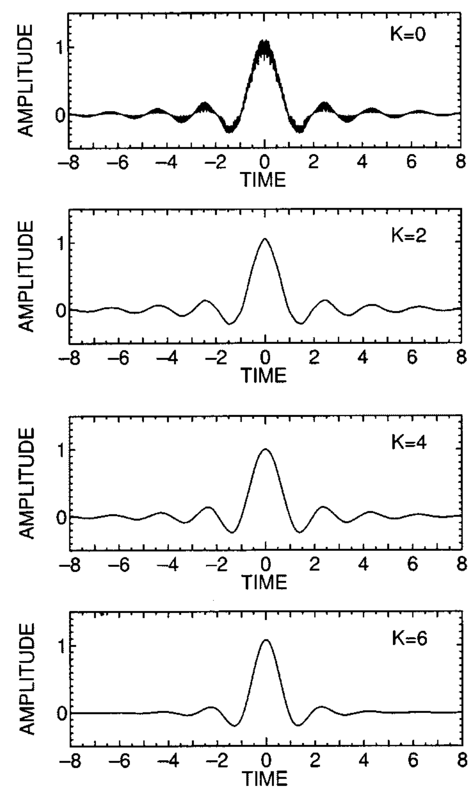

Fig. 8. Scaling functions in Example 2.

V. DESIGN EXAMPLES

In this section, we will use the design method proposed in this paper to design the WSS paraunitary filterbanks composed of a single complex allpass filter and present some numerical examples to demonstrate the effectiveness of the proposed method.

Example 1: We consider the design of the maximally flat filters. The filter coefficients of $A(z)$ with the maximum flatness can be calculated from (27) and ones of $N = 2, 4, 6, 8$ are listed in Table I. Note that when M is even, i.e., $N = 4, 8, \ldots$, then $\eta = \pi/4$, and when M is odd, i.e., $N = 2, 6, \ldots$, then $\eta = -3\pi/4$. The resulting phase responses of $A(z)$ are shown in Fig. 2, and the magnitude responses of $H(z)$ and

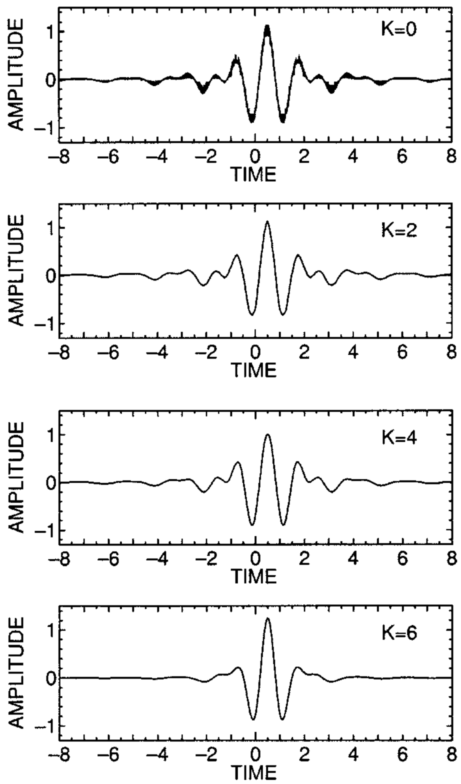

Fig. 9. Wavelet functions in Example 2.

$G(z)$ are shown in Fig. 3, respectively. It is clear that the frequency responses become more flat as N increases. The scaling and wavelet functions generated by the above paraunitary filterbanks are shown in Figs. 4 and 5, respectively. It can be seen in Figs. 4 and 5 that both the scaling and wavelet functions are symmetric and are more regular with an increasing N.

Example 2: We consider the design of the paraunitary filterbanks with a given degrees of flatness. The order of $A(z)$ is $N = 6$. The cut-off frequency is $\omega_p = 0.45\pi$, and $\eta = -3\pi/4$. We have designed $A(z)$ with $K = 0, 2, 4, 6$ by using the proposed method. The obtained filter coefficients are listed in Table II. The resulting phase responses of $A(z)$ are shown in Fig. 6, and the magnitude responses of $H(z)$ and $G(z)$ are shown in Fig. 7, respectively. It can be seen that $K = 6$ corresponds to the maximally flat solution, and $K = 0$ is the minimax solution that has no zero located at $z = 1$ and $z = -1$. The magnitude error increases as K increases. The generated scaling and wavelet functions are shown in Figs. 8 and 9, respectively. It is seen in Figs. 8 and 9 that when $K = 0$, the scaling and wavelet functions are not continuous because the regularity conditions are not satisfied. The scaling and wavelet functions become more smooth with an increasing K.

VI. Conclusion

In this paper, we have discussed the design of the WSS paraunitary filterbanks composed of a single complex allpass filter and given a new class of real-valued orthonormal symmetric wavelet bases. From the symmetry and orthonormality conditions of wavelets, we have first given the conditions imposed on the complex allpass filter and derived the complex allpass filters transfer function to satisfy these conditions. Second, we have proposed a new method for designing the WSS paraunitary filterbanks with a given degrees of flatness from the viewpoints of the regularity and frequency selectivity. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez exchange algorithm. Therefore, the filter coefficients can be easily obtained by solving the eigenvalue problem to find the positive minimum eigenvalue, and the optimal solution is attained through a few iterations. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step. Furthermore, both the maximally flat and minimax solutions are also included in the proposed method as two specific cases. The maximally flat filters have a closed-form solution without any iteration. Finally, some design examples are presented to demonstrate the effectiveness of the proposed method.

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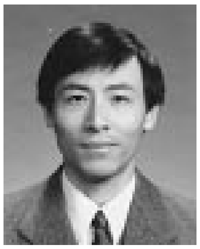

Xi Zhang (M'94–SM'01) received the B.E. degree in electronics engineering from the Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 1984 and the M.E. and Ph.D. degrees in communication engineering from the University of Electro-Communications (UEC), Tokyo, Japan, in 1990 and 1993, respectively.

He was with the Department of Electronics Engineering at NUAA from 1984 to 1987 and the Department of Communications and Systems at UEC from 1993 to 1996, all as an Assistant Professor. Currently, he is with the Department of Electrical Engineering, Nagaoka University of Technology, Niigata, Japan, as an Associate Professor. His research interests are in the areas of digital signal processing, filter design theory, filterbanks and wavelets, and their applications to image coding.

Dr. Zhang is a member of the IEICE of Japan. He received the Award of Science and Technology Progress of China in 1987.

Akira Kato received the B.E. degree in electrical engineering from Nagaoka University of Technology (NUT), Niigata, Japan, in 1998. He is currently pursuing the M.E. degree at NUT. His research interest is digital signal processing.

Mr. Kato is a student member of the IEICE of Japan.

Toshinori Yoshikawa (M'89) received the B.E., M.E. and Dr. Eng. degrees from Tokyo Institute of Technology, Tokyo, Japan, in 1971, 1973, and 1976, respectively.

From 1976 to 1983, he was with Saitama University, Saitama, Japan, where he was involved with research on signal processing and its software development. Since 1983, he has been with Nagaoka University of Technology, Niigata, Japan, where he is currently a Professor. His main research area is digital signal processing.

Dr. Yoshikawa is a member of the IEICE of Japan.