Derivative-Controlled Design of Linear-Phase FIR Filters via Waveform Moments

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***Abstract*—A new method for designing linear-phase finite impulse response (FIR) filters is proposed by using the blockwise waveform moments. The proposed method yields linear-phase FIR filters whose magnitude response and its derivatives to a certain order take the prescribed values at the equally spaced frequency points. The design procedure only needs to solve a system of linear equations, whose size is slightly smaller than the degree of the resulting filter. In addition, the inversion of the linear equations can be essentially precomputed. Therefore, the proposed design method is computationally efficient. In particular, for some important cases, i.e., the maximally flat R-regular Lth-band FIR filters, a closed-form formula can be obtained. It is also shown that the resulting R-regular Lth-band FIR filters have the zero intersymbol interference property.**

***Index Terms*—Linear-phase FIR filter, maximally flat filter, Nyquist filter, waveform moment.**

I. Introduction

LIKE their well-known statistical counterparts, waveform moments describe some of the important geometric characteristics of a signal, e.g., the position of center, symmetry, etc. By its definition, a waveform moment can be easily calculated from the signal. For these reasons, they have been used to describe the characteristics of the waveform of a signal or the impulse response of a digital filter in more or less explicit manners. Some applications are: the coefficient sensitivity analysis of FIR filters [23], [24], deconvolution [28], cepstrum analysis [26], characterization of Coiflets [22], and the derivative analysis of the squared-magnitude of FIR filters [8]. The waveform moments have been extended to the blockwise waveform moments [25]. The extension enables us to grasp the periodical characteristics of the signal or impulse response analyzed. Specifically, it has been shown in [25] that the waveform moments of a discrete signal exactly describe certain derivative behaviors (the value, tangency, curvature, …) of the frequency response at the equally spaced frequency points and has been used to evaluate the quality of signals recovered by the interpolation filters [27]. This property suggests to us the possibility of a general design method of FIR filters such that the derivative behaviors of the frequency response can be controlled arbitrarily. Maximally flat FIR filters are an important class of digital filters and have been well studied in [2]–[8]. In addition, R-regular Lth-band filters play an important role in constructing L band wavelet filterbanks [9]. Lth-band filters are required to have the Nyquist property, i.e, the zero intersymbol interference property. The design methods of these filters have been proposed in [9]–[21]. In [9] and [10], the closed-form design of R-regular Lth-band filters are presented, but the case of even R is considered only. In [21], one method using the spectral mask constraints is proposed for designing general FIR filters. Although one can control arbitrary behavior of the frequency response by placing an appropriate mask, in [21], it was not discussed how to design the maximally flat R-regular Lth-band FIR filters.

In this paper, we propose a *derivative-controlled* design method of linear-phase FIR filters using the blockwise waveform moments. We consider all four types of linear-phase FIR filters. Given the values of the 0th to $(R-1)$th derivatives of the magnitude response at L-division frequency points, a set of filter coefficients can be easily obtained by the proposed method. The design procedure consists of two stages. First, the RL specified derivatives are transformed into the RL blockwise moments. This stage involves R times of L-point DFT. Second, the filter coefficients are calculated from the RL blockwise moments by solving L systems of Vandermonde equations of size R by R. The length of the resulting FIR filters is approximately RL and exactly determined by L, R, and the type of the filters. We note that the Vandermonde matrices to be inverted are actually *constant for the specification*, and then, the inversion can be precomputed. Therefore, the design procedure is computationally efficient. In particular, for the maximally flat filters, i.e., in the case when all the derivatives other than 0th are required to be zero, the design procedure is so simplified that a closed-form formula can be obtained. The closed-form formula is presented for the maximally flat R-regular Lth-band FIR filters, including the case of odd R that was not covered in [9].

One may think at this point that we do not have many reasons to force the derivatives of the magnitude response at grid points of the frequency to be certain values other than zero. For example, the tangency is essential at least for digital differentiators. Consider the following situation. Suppose that we have a many number of signals (say, images), which fall into a certain class (say, of human faces), and we wish to extract some useful partial informations from these signals by means of a linear time invariant (LTI) system. In many cases, the characterization of the class of signals and the partial information that we wish to extract depend heavily on the intuition of human, and an *a priori*

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mathematical characterization of them is usually unavailable. In such a case, we have to repeat tuning the parameters of the LTI and inspecting the output of the LTI until the LTI can correctly extract the partial information. The LTI parameters supplied by us should have a clear intuitive meaning. On the other hand, the parameters also should have a concrete mathematical sense; otherwise, the further fine theoretical analysis of the class of the signals is impossible, even after the LTI outputs satisfactory results for a set of examples. We believe that the derivatives of the magnitude are suitable in both senses, and the proposed method considerably reduces the turn-around time.

The paper is organized as follows. In Section II, we review the blockwise waveform moments and their relationship to the derivatives of the frequency response. In Section III, we focus on the case of linear-phase FIR filters and describe the relationship between the blockwise waveform moments and the derivatives of the magnitude response. Section IV presents the design procedure of linear-phase FIR filters. In Section V, the closed-form formula for the maximally flat R-regular Lth-band FIR filters is derived. In Section VI, some design examples are presented to demonstrate the effectiveness of the proposed design method.

II. Definition of Blockwise Waveform Moment

Let $H(z)$ be the transfer function of an FIR filter of degree N, and let $h[i]$ be its real-valued impulse response

$$H(z) = \sum_{i=0}^{N} h[i] z^{-i}. \quad (1)$$

For a non-negative integer r, the rth waveform moment (around zero) for the impulse response $h[i]$ is defined by

$$m_r = \sum_{i=0}^{N} i^r h[i], \quad (r \in \mathbb{Z}_{\geq 0}) \quad (2)$$

which is an analog of the statistical moment, if one sees $h[i]$ as a discrete probabilistic distribution. Like the statistical counterpart, it is useful to describe the geometric characteristics of the impulse response $h[i]$. Especially, the center μ of $h[i]$ is defined in terms of the waveform moment m_r as

$$\mu = \frac{m_{\alpha+1}}{(\alpha+1)m_\alpha} \quad (3)$$

where α is the smallest non-negative integer for which $m_\alpha \neq 0$ holds. When $h[i]$ is symmetrical or antisymmetrical (i.e., $h[i] = \pm h[N-i]$ $(0 \leq i \leq N)$), μ is consistent with the natural center $N/2$. Then, the rth waveform moment around the center μ is defined by

$$\mu_r = \sum_{i=0}^{N} (i-\mu)^r h[i] \quad (r \in \mathbb{Z}_{\geq 0}). \quad (4)$$

The waveform moment has a definite sense in the frequency domain, as well as in the time domain. Specifically, the rth waveform moment m_r describes the rth differential coefficient of the frequency response of the corresponding FIR filter at the origin

$$m_r = j^r \left. \frac{d^r H(e^{j\omega})}{d\omega^r} \right|_{\omega=0}. \quad (5)$$

An extended version of the waveform moments, called blockwise waveform moments, has been proposed in [25], which enables us to describe a periodic property of the impulse response. Let L be a positive integer. Then, the extended blockwise waveform moment around the origin is defined by

$$\begin{aligned} m_r(n,L) &= \sum_{\substack{i=0 \\ i \equiv n \bmod L}}^{N} i^r h[i] \\ &= \sum_{i=0}^{M_n} (Li+n)^r h[Li+n] \quad (r \in \mathbb{Z}_{\geq 0}) \end{aligned} \quad (6)$$

where $0 \leq n \leq L-1$, and

$$M_n = \left\lfloor \frac{N-n}{L} \right\rfloor. \quad (7)$$

Note that the waveform moment m_r in (2) can be decomposed into the sum of L blockwise waveform moments $m_r(n,L)$:

$$\begin{aligned} m_r &= \sum_{n=0}^{L-1} \sum_{i=0}^{M_n} (Li+n)^r h[Li+n] \\ &= \sum_{n=0}^{L-1} m_r(n,L) \end{aligned} \quad (8)$$

which corresponds to the following polyphase decomposition:

$$\begin{aligned} H(z) &= \sum_{n=0}^{L-1} z^{-n} \sum_{i=0}^{M_n} h[Li+n] z^{-Li} \\ &= \sum_{n=0}^{L-1} z^{-n} H_n(z^L). \end{aligned} \quad (9)$$

It follows from the definition of $m_r(n,L)$ in (6) that

$$\left. \frac{d^r H(e^{j\omega})}{d\omega^r} \right|_{\omega=(2\pi k/L)} = \frac{1}{j^r} \sum_{n=0}^{L-1} m_r(n,L) e^{-j(2\pi kn/L)} \ (0 \leq k \leq L-1) \quad (10)$$

i.e., the blockwise waveform moments describe the derivative behavior of the frequency response at L frequency points $\omega = 2\pi k/L$ $(0 \leq k \leq L-1)$. It is clear that the r^{th} derivatives of the frequency response at $\omega = 2\pi k/L$ $(0 \leq k \leq L-1)$ are the L-point discrete fourier transform (DFT) of the blockwise waveform moments $m_r(n,L)$ $(0 \leq n \leq L-1)$. Thus, we have by the inverse transform

$$m_r(n,L) = \frac{j^r}{L} \sum_{k=0}^{L-1} \left. \frac{d^r H(e^{j\omega})}{d\omega^r} \right|_{\omega=(2\pi k/L)} e^{j(2\pi kn/L)} \quad (0 \leq n \leq L-1). \quad (11)$$

By (6) and (11), the blockwise waveform moments $m_r(n,L)$ bridge between the time and frequency domains. The blockwise waveform moments around the center can also be defined in the same manner

$$\mu_r(n,L) = \sum_{i=0}^{M_n} (Li+n-\mu)^r h[Li+n] \quad (r \in \mathbb{Z}_{\geq 0}, \ 0 \leq n \leq L-1) \quad (12)$$

where μ is the center of the impulse response $h[i]$ defined by (3). Similarly, the following decomposition holds:

$$\mu_r = \sum_{n=0}^{L-1} \mu_r(n, L). \tag{13}$$

In the following section, we will examine the relationship between $\mu_r(n, L)$ and the magnitude response of linear-phase FIR filters.

III. Linear-Phase FIR Filters

Hereafter, we concentrate on the case of linear-phase FIR filters. There are four types of linear-phase FIR filters, depending on the parity of the filter degree N and the symmetry of the impulse response. A linear-phase FIR filter is referred to as "Type $\nu + 2q + 1$" in [1], if we define parameters $\nu, q \in \{0, 1\}$ by

$$\nu = \begin{cases} 0, & (N\text{: even}) \\ 1, & (N\text{: odd}) \end{cases} \tag{14}$$

$$q : h[N-i] = (-1)^q h[i] \quad (0 \le i \le N) \tag{15}$$

where in any case, the center defined by (3) is

$$\mu = \frac{N}{2}. \tag{16}$$

When $H(z)$ is the transfer function of a linear-phase FIR filter

$$A(\omega) = H(e^{j\omega}) e^{j((N/2)\omega - (\pi/2)q)} = j^{-q} \sum_{i=0}^{N} h[i] e^{-j(i-(N/2))\omega} \tag{17}$$

is a real function of ω and represents the signed magnitude response of the filter.

In the previous section, we have seen that the blockwise waveform moments $m_r(n, L)$ are related to the rth derivatives of the frequency response $H(e^{j\omega})$ at the L-division frequency points $\omega = 2\pi k/L$ $(0 \le k < L)$ via L-point DFT in (10) and (11). For the linear-phase FIR filters, there is the same relation between the blockwise waveform moments around the center $\mu_r(n, L)$ and the rth derivatives of the magnitude response $A(\omega)$. By derivating r times (17) by ω and substituting $2\pi k/L$ for ω, we get

$$A^{(r)}\left(\frac{2\pi k}{L}\right) = j^{-(r+q)} \sum_{n=0}^{L-1} \mu_r(n, L) e^{-j(2\pi k/L)(n-(N/2))} \quad (0 \le k \le L-1) \tag{18}$$

where $A^{(r)}(2\pi k/L)$ denotes $(d^r/d\omega^r)A(\omega)\big|_{\omega=2\pi k/L}$. That is, $A^{(r)}(2\pi k/L)$ is L-point DFT of $\mu_r(n, L)$. By IDFT, we have

$$\mu_r(n, L) = \frac{j^{r+q}}{L} \sum_{k=0}^{L-1} A^{(r)}\left(\frac{2\pi k}{L}\right) e^{j(2\pi k/L)(n-(N/2))} \quad (0 \le n \le L-1). \tag{19}$$

Given the rth derivatives of the magnitude response at the frequency points $\omega = 2\pi k/L$ $(0 \le k \le L-1)$, the rth blockwise waveform moments $\mu_r(n, L)$ $(0 \le n \le L-1)$ can be calculated directly via (19). On the other hand, the rth blockwise waveform moments are related to the impulse response by

$$\mu_r(n, L) = \sum_{i=0}^{M_n} \left(Li + n - \frac{N}{2}\right)^r h[Li + n] \quad (0 \le n \le L-1). \tag{20}$$

These properties of the blockwise waveform moments suggest to us a design procedure for linear-phase FIR filters, which is able to control the derivative behavior of the magnitude response at the frequency points $\omega = 2\pi k/L$.

IV. Design Procedure for Linear-Phase FIR Filters

In this section, we present a design method for linear-phase FIR filters. Based on the properties of the blockwise waveform moments in the previous section, FIR filters are designed in a *derivative-controlled* manner. That is, the magnitude response and its derivatives (of order $< R$) of the filter take the prescribed values at the frequency points $\omega = 2\pi k/L$ $(0 \le k \le L-1)$.

A. Overview

The design procedure has two stages. First, the blockwise waveform moments are calculated from the derivatives via the IDFT in (19). Then, the filter coefficients are obtained by solving the linear equations in (20).

B. Specifications and Conditions

Input:

a) L: the number of division of the block;
b) R: the number of derivatives controlled, i.e., 0th to $(R-1)$th derivatives of the magnitude to be controlled;
c) ν, q: Zero-One parameters defining the filter type as in (14) and (15); the resulting filter is of Type $\nu + 2q + 1$;
d) $(a_{r,k})_{0 \le r < R,\ 0 \le k < L}$: real matrix of size R by L; $a_{r,k}$ representing $A^{(r)}(2\pi k/L)$, which is the desired value of the rth derivative of the magnitude at $\omega = 2\pi k/L$. It must be satisfied that

$$a_{r,L-k} = (-1)^{\nu+q+r} a_{r,k} \quad (0 \le k < L,\ 0 \le r \le R-1). \tag{21}$$

■

The above condition on the matrix $(a_{r,k})$ comes from the fact that the filters of Type $\nu + 2q + 1$ are constrained by $A^{(r)}(2\pi - \omega) = (-1)^{\nu+q+r} A^{(r)}(\omega)$ from (17). In addition, $a_{r,0}$ is constrained by both $A^{(r)}(0) = (-1)^{\nu+q+r} A^{(r)}(2\pi)$ and $A^{(r)}(0) = (-1)^{\nu} A^{(r)}(2\pi)$. For these reasons, we are able to specify only about half $a_{r,k}$'s, and some entries must be zero. To be precise, writing

$$\lambda = \begin{cases} 0, & (L\text{: even}) \\ 1, & (L\text{: odd}) \end{cases} \tag{22}$$

$$\rho = \begin{cases} 0, & (R\text{: even}) \\ 1, & (R\text{: odd}) \end{cases} \tag{23}$$

respectively, the number of free entries in the matrix is counted as

$$\frac{LR + (2(1-\lambda)(1-\nu) + \lambda)(-1)^q \rho}{2} \tag{24}$$

TABLE I
FILTER DEGREE N

	L : even ($\lambda = 0$)		L : odd ($\lambda = 1$)	
	R : even ($\rho = 0$)	R : odd ($\rho = 1$)	R : even ($\rho = 0$)	R : odd ($\rho = 1$)
Type 1 ($\nu = 0, q = 0$)	$LR-2$	LR	$LR-2$	$LR-1$
Type 2 ($\nu = 1, q = 0$)	$LR-1$	$LR-1$	$LR-1$	LR
Type 3 ($\nu = 0, q = 1$)	LR	$LR-2$	LR	$LR-1$
Type 4 ($\nu = 1, q = 1$)	$LR-1$	$LR-1$	$LR-1$	$LR-2$

and it is required that

$$a_{r,0} = 0 \quad \text{for } r \text{ such that } r+q \text{ is odd.} \tag{25}$$

In addition, if L is even

$$a_{r,L/2} = 0 \quad \text{for } r \text{ such that } \nu+q+r \text{ is odd.} \tag{26}$$

It is always possible to remove redundancies from $(a_{r,k})$ by restricting the range of k appropriately; however, this will introduce some complication in notation.

Output: $(h[i])_{0 \le i \le N}$: The coefficients of linear-phase FIR filters of Type $\nu + 2q + 1$, whose magnitude $A(\omega)$ satisfies

$$A^{(r)}\left(\frac{2\pi k}{L}\right) = a_{r,k} \quad (0 \le k \le L-1,\ 0 \le r \le R-1).$$

Note that

$$A^{(r)}(0) = 0 \quad \text{for } r \text{ such that } r+q \text{ is odd} \tag{27}$$

must be satisfied from (25) and when L is even, from (26)

$$A^{(r)}(\pi) = 0 \quad \text{for } r \text{ such that } \nu+q+r \text{ is odd.} \tag{28}$$

The filter degree N is written as

$$\begin{aligned} N =& LR \\ &+ (2(1-\lambda)(1-\nu)+\lambda)(-1)^q \rho \\ &+ (2q-1)(1-\nu) - 1. \end{aligned} \tag{29}$$

■

The filter degree N is of the possible minimum since the real vectors $(h[i])_{0 \le i \le N}$ with the condition of (15) span a vector space of dimension

$$\frac{N+\nu}{2} + (1-\nu)(1-q)$$

which should agree with (24), which is the dimension of the real vector space spanned by the matrices $(a_{r,k})$ satisfying (21). Once L and R are fixed, N takes only three values LR, $LR-1$, and $LR-2$. The filter degree N for all cases is shown in Table I.

C. Calculating Blockwise Waveform Moments

Using the IDFT relation in (19), the blockwise waveform moments $\mu_r(n,L)$ are calculated from the specification $a_{r,k} = A^{(r)}(2\pi k/L)$. That is

$$\mu_r(n,L) = \frac{j^{r+q}}{L} \sum_{k=0}^{L-1} a_{r,k}\, e^{j(2\pi k/L)(n-(N/2))} \quad (0 \le n \le L-1,\ 0 \le r \le R-1) \tag{30}$$

which need R times of L-point IDFT.

D. Solving the Defining Linear System

Now, we have the value of $\mu_r(n,L)$ for $0 \le n \le L-1$ and $0 \le r \le R-1$. From (20), which is the definition of the blockwise waveform moments around the center, we have a system of linear equations

$$\sum_{i=0}^{M_n} \left(Li + n - \frac{N}{2}\right)^r h[Li+n] = \mu_r(n,L) \quad (0 \le r \le R-1). \tag{31}$$

When $M_n = R-1$, the system becomes the nondegenerate Vandermonde matrix of size R by R

$$\left(v_{r,i} = \left(Li + n - \frac{N}{2}\right)^r\right)_{0 \le r,i \le R-1}.$$

Thus, solving the system for the values of $\mu_r(n,L)$ $(0 \le r \le R-1)$ gives a set of filter coefficients $h[Li+n]$ $(0 \le i \le M_n)$ immediately. The condition $M_n = M-1$ is true for almost all n since $N = LR - c$ ($c = 0$, 1, 2) from Table I and $M_n = \lfloor (N-n)/L \rfloor = \lfloor R - (c+n)/L \rfloor$ by (7).

The first exception is $M_n = R$ only when $c = 0$ and $n = 0$. From Table I, it occurs only if $q = 0$, $\rho = 1$, and $\lambda = \nu$, or $q = 1$, $\rho = 0$, and $\nu = 0$. For the case $q = 0$, $\rho = 1$, and $\lambda = \nu$, plugging the values of $n = 0$, $N = LR$ with odd R and $q = 0$ into (30), the blockwise waveform moment $\mu_r(0,L)$ is in fact

$$\mu_r(0,L) = \frac{j^r}{L} \sum_{k=0}^{L-1} a_{r,k}(-1)^k.$$

Using (21), it can be rewritten as

$$\mu_r(0,L) = \frac{j^r}{L}\left((-1)^r a_0 + (-1)^{\nu+\lambda+r} \sum_{k=1}^{L-1} a_{r,k}(-1)^k\right).$$

Thus, $\mu_r(0,L)$ vanishes for odd r since $\lambda + \nu$ is even. On the other hand, from the facts that $h[Li] = h[N-Li] = h[L(R-i)]$ and that R is odd, the left-hand side of (31) is 0 for odd r and $2\sum_{i=0}^{(R-1)/2} (Li - N/2)^r$ for even r. Therefore, (31) shrinks to

$$2 \sum_{i=0}^{(R-1)/2} \left(Li - \frac{N}{2}\right)^{2r} h[Li] = \mu_{2r}(0,L) \quad \left(0 \le r \le \frac{R-1}{2}\right)$$

which is a nondegenerate Vandermonde system of size $(R+1)/2$ by $(R+1)/2$[1] and can be solved to obtain $h[Li]$ for $0 \le$

[1]No two columns are the same because $Li - N/2 = -(Li' - N/2)$ implies $L(i+i') = N = LR$, which is impossible if both i and i' are less than $R/2$.

$i \le (R-1)/2$. The relation $h[Li] = h[L(R-i)]$ determines $h[Li]$ for $(R+1)/2 \le i \le R = M_0$.

For the case $q = 1$, $\rho = 0$, and $\nu = 0$, by a similar argument, (31) can be reduced to

$$2\sum_{i=0}^{R/2-1}\left(Li - \frac{N}{2}\right)^{2r+1} h[Li] = \mu_{2r+1}(0, L) \qquad \left(0 \le r \le \frac{R}{2} - 1\right)$$

whose matrix is of Vandermonde type, and $h[Li] = -h[L(R-i)]$ gives $h[Li]$ for $R/2 < i \le R = M_0$.[2]

Another exception is $M_n = R - 2$ only when $N = LR - 2$ and $n = L - 1$. From Table I, it occurs if and only if $\rho = \nu = q = 0$ or $\rho = q = 1$, $\lambda = \nu$. When $\rho = \nu = q = 0$, the system is reduced to

$$2\sum_{i=0}^{R/2-2}\left(Li + L - 1 - \frac{N}{2}\right)^{2r} h[Li + L - 1] + \delta(r) h\left[\frac{N}{2}\right] = \mu_{2r}(L-1, L) \qquad \left(0 \le r \le \frac{R}{2} - 1\right)$$

where $\delta(r) = 1$ if $r = 0$; otherwise, $\delta(r) = 0$. When $\rho = q = 1$, $\lambda = \nu$, the system is reduced to

$$2\sum_{i=0}^{(R-3)/2}\left(Li + L - 1 - \frac{N}{2}\right)^{2r+1} h[Li + L - 1] = \mu_{2r+1}(L-1, L) \qquad \left(0 \le r \le \frac{R-3}{2}\right).$$

In either case, $h[Li + n]$ for $0 \le i \le M_n$ are obtained by solving a linear system of at most size R by R. We can obtain all filter coefficients by solving such a linear system L times.

E. Computational Complexity and Numerical Stability

The overall procedure needs to solve a linear system of N variables, and so, it would take $O(N^3)$ multiplications *a priori*. However, the blockwise waveform moments localize the problem and reduce its complexity. The first stage takes R times of L-point IDFT, which is accomplished by $O(RL^2)$ multiplications. If L is a power of 2, then FFT can be employed; thus, it is reduced to $O(RL \log L)$. The second stage needs to solve L times the Vandermonde system of size R by R.

An algorithm has been presented in [30, Sec.2.8], which exploits symbolic polynomial multiplications to solve a Vandermonde system of size R by R in $O(R^2)$ scalar multiplications. Thus, the total number of multiplications in this stage is $O(LR^2)$. Therefore, the proposed design procedure takes only $O(N \max(L, R))$ multiplications.

The numerical stability of the design procedure depends on R and L. The condition number of the matrix in (31) may be big when R and L are large. By our experiments, for all combinations of $L \le 12$, $R \le 8$ floating-point operations with nine-decimal-digits precision guarantee the upper bound 10^{-8} on relative error of each entry of the inverse matrix. The error bound was estimated using the GP/PARI system [29], which can invert the matrix without error by the exact integer or rational-number operations. When (L, R) is out of the above range, we should pay an attention to the implementation of the algorithm. We may use the exact integer operations instead of the floating point operations to avoid numerical difficulties. If the coefficients of the master polynomial, at the heart of the algorithm, are calculated by the exact integer operations of GP/PARI, then the 10^{-8} relative error bound is satisfied, even when (L, R) =(256,4), (32,32), or (4,256) if the scalars other than the polynomial coefficients are nine-decimal-digits precision floating-point numbers. This implementation of the algorithm considerably increases the running time (in this case, the running time cannot be bounded only by the number of multiplications since the largest magnitude of scalars that appear in the computation is about $L^R((R/2)!)^2$, which requires a bit length of $R \log_2 LR$). Despite this fact, the use of the exact operations is practical. An Athlon 700-MHz personal computer with GP/PARI calculator finishes the whole design procedure in 12.03 [s], 2.69 [s], and 12.63 [s] for the cases (L, R) =(256,4), (32,32), and (4,256), respectively.

F. Precomputation and Superposition

We also note that every matrix to be inverted is constant once the parameters L, R, ν, and q are fixed, and thus, these inversions can be precomputed. In other words, for each pair (r_0, k_0) in the range $0 \le r_0 \le R-1$, $0 \le k_0 \le L-1$, we can precompute the filter coefficients h_{r_0,k_0} corresponding to the specification

$$a_{r,k} = \begin{cases} 1, & (k = k_0, r = r_0) \\ (-1)^{\nu+q+r}, & (k = L - k_0, r = r_0) \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Let these h_{r_0,k_0} be the prototype filters. Note that for such a sparse specification, $\mu_r(n, L)$ is zero or just a special value of a sinusoidal fuction multiplied by an elementary factor (33), shown at the bottom of the page. Given the specification matrix $(a_{r,k})$, the corresponding filter can be designed by superposition of these prototype filters with obvious factors $a_{r,k}$. This implies that the design procedure can respond quickly to the modification on the specification matrix $(a_{r,k})$.

[2] Note that $h[L(R/2)] = -h[L(R/2)] = 0$.

$$\mu_r(n, L) = \begin{cases} \frac{(2-\delta(k_0))(-1)^{(r_0+q)/2}}{L}\cos\left(\frac{2\pi k_0\left(n-\frac{N}{2}\right)}{L}\right), & (r = r_0,\ r_0 + q\text{:even}) \\ \frac{(2-\delta(k_0))(-1)^{(r_0+q+1)/2}}{L}\sin\left(\frac{2\pi k_0\left(n-\frac{N}{2}\right)}{L}\right), & (r = r_0,\ r_0 + q\text{: odd}) \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

V. Closed-Form Formula of L^{th}-Band FIR Filters

Maximally flat FIR filters with linear phase are known to be useful in many applications. For constructing a L-band regular wavelet filterbank, the design of lowpass scaling filters is critical [9]. In [9], the scaling filter $G(z)$ is constructed by first designing a R-regular Lth-band filter $H(z)$ and then decomposing it as $H(z)=G(z)G(z^{-1})$, where $H(z)$ is a Type 1 linear-phase FIR filter, and R is restricted to an even number for orthonormal wavelet filterbanks. They presented two closed-form formulae for $H(z)$ with an even R. Another closed-form formula for the same $H(z)$ is also presented in [10]. For biorthogonal wavelets, however, the regularity R is not necessarily an even number, i.e., R may be odd number. The design of $H(z)$ with an odd R is not still discussed.

Our derivative-controlled design is applicable to this special (but useful) class of filters by setting all the magnitude derivatives of positive order to be 0 at every L-division frequency points. In this case, the linear equations to be solved are so simplified that the filter coefficients can be obtained in a closed-form formula. Our closed-form formula covers all four types of linear-phase FIR filters and is applicable whenever the regularity R is odd or even; thus, it includes the results presented in [9] and [10].

Suppose that all the entries of the specification matrix $(a_{r,k})$ are zero, except for the row $r=0$. Then, every $\mu_r(n,L)$ vanishes for $r>0$, and the application of the Cramer's formula to (31) (and each of its reduced forms) yields the presentation of the filter coefficient as a quotient of two Vandermonde's determinants, which is simplified to (34), shown at the bottom of the page. Note also that from the argument in Section IV-F, once the filter coefficients are explicitly calculated using (33) and (34) for at most $(L+\lambda)/2$ possible prototype filters having the specification shown in (32) with $r_0=0$, then others can be obtained by superposition of these prototype filters. Furthermore, the resulting filter is a Nyquist filter if the filter degree N is even, that is

$$h[Li+n]=0$$
$$\left(0\le i\le M_n,\ n\equiv\frac{N}{2}\ \mathrm{mod}\ L,\ Li+n\ne\frac{N}{2}\right)\qquad(35)$$

independent of the specification $a_{0,k}=A(2\pi k/L)$. It is because the constant factor $\prod_{s\ne i}(Ls+n-N/2)$ vanishes if $n\equiv N/2 \bmod L$ and $Li+n\ne N/2$. Note that $n\equiv N/2 \bmod L$ eliminates the possibility of the second line of (34), since $N/2\not\equiv 0 \bmod L$ if R is odd and $N=LR$. Thus, these maximally flat FIR filters possess the zero intersymbol interference property.

TABLE II
Filter Coefficients for Type 1 Lowpass Filter With $L=4$, $R=8$

$h[0]$ =	-0.000410	$=h[30]$
$h[1]$ =	-0.000620	$=h[29]$
$h[2]$ =	-0.000480	$=h[28]$
$h[3]$ =	0.000000	$=h[27]$
$h[4]$ =	0.003900	$=h[26]$
$h[5]$ =	0.005980	$=h[25]$
$h[6]$ =	0.004770	$=h[24]$
$h[7]$ =	0.000000	$=h[23]$
$h[8]$ =	-0.018420	$=h[22]$
$h[9]$ =	-0.029910	$=h[21]$
$h[10]$ =	-0.025780	$=h[20]$
$h[11]$ =	0.000000	$=h[19]$
$h[12]$ =	0.071590	$=h[18]$
$h[13]$ =	0.149530	$=h[17]$
$h[14]$ =	0.214790	$=h[16]$
$h[15]$ =	0.250000	$=h[15]$

VI. Design Examples

Although the design procedure can be applied to the design of general linear-phase FIR filters, a few design examples are shown to illustrate the properties of the resulting filters. The first two examples are frequency selective filters of the maximally flat type. The third is a differentiator, for which the tangency is specified. The design procedure devotes all of its degrees of freedom to the control on the L-division frequency points; thus, the magnitude and its derivatives are exactly controlled on these points.

A. Example 1 (Type 1 Lowpass Filters)

Type 1 maximally flat $L=4$th-band lowpass filters are designed for $R=$4, 8, and 16 by using (34) under the specification

$$a_{r,k}=A^{(r)}\left(\frac{2\pi k}{4}\right)=\begin{cases}1, & (r=0,\ k=0)\\ 0, & \text{(otherwise)}.\end{cases}$$

The filter degrees for $R=4$, 8, and 16 are 14, 30, and 62, respectively. As shown in the previous section, they are Nyquist filters. The filter coefficients for $R=8$ are shown in Table II, in which we see that $h[3]=h[7]=h[11]=h[19]=h[23]=h[27]=0$. The magnitude responses are shown in Fig. 1. They exhibit a flat behavior at $\omega=0,\pi/2,\pi$ and exhibit a deviation from zero,

$$h[Li+n]=\begin{cases}\dfrac{(-1)^i\prod\limits_{\substack{0\le s\le R-1\\ s\ne i}}(Ls+n-\frac{N}{2})}{L^{R-1}i!(R-i-1)!}\mu_0(n,L), & (M_n=R-1)\\[2ex] \dfrac{(-1)^i\prod\limits_{\substack{0\le s\le (R-1)/2\\ s\ne i}}(s-\frac{R}{2})^2}{2i!(R-i)!}\mu_0(0,L), & (n=0,q=0,\rho=1,\lambda=\nu)\\[2ex] \mu_0(L-1,L), & (n=L-1,Li+(L-1)=\frac{N}{2},\ q=\rho=\nu=0)\\ 0, & \text{(otherwise)}.\end{cases}\qquad(34)$$

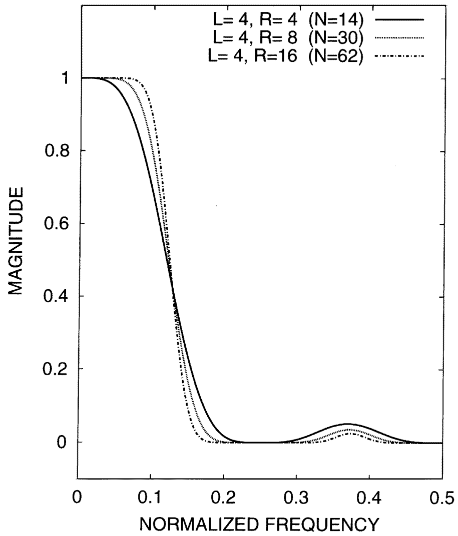

Fig. 1. Magnitude responses of Type 1 lowpass filters.

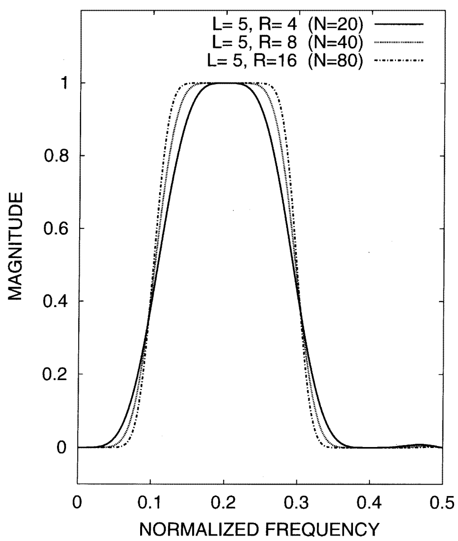

Fig. 2. Magnitude responses of Type 3 bandpass filters.

TABLE III
FILTER COEFFICIENTS FOR TYPE 3 BANDPASS FILTER WITH $L = 5$, $R = 8$

$h[\ 0]$ =	0.000000	$= -h[40]$
$h[\ 1]$ =	0.000514	$= -h[39]$
$h[\ 2]$ =	0.000532	$= -h[38]$
$h[\ 3]$ =	-0.000563	$= -h[37]$
$h[\ 4]$ =	-0.000611	$= -h[36]$
$h[\ 5]$ =	0.000000	$= -h[35]$
$h[\ 6]$ =	-0.004885	$= -h[34]$
$h[\ 7]$ =	-0.005157	$= -h[33]$
$h[\ 8]$ =	0.005587	$= -h[32]$
$h[\ 9]$ =	0.006217	$= -h[31]$
$h[10]$ =	0.000000	$= -h[30]$
$h[11]$ =	0.022797	$= -h[29]$
$h[12]$ =	0.025140	$= -h[28]$
$h[13]$ =	-0.028732	$= -h[27]$
$h[14]$ =	-0.034195	$= -h[26]$
$h[15]$ =	0.000000	$= -h[25]$
$h[16]$ =	-0.085487	$= -h[24]$
$h[17]$ =	-0.111735	$= -h[23]$
$h[18]$ =	0.167603	$= -h[22]$
$h[19]$ =	0.341949	$= -h[21]$
$h[20]$ =	0.000000	$= -h[20]$

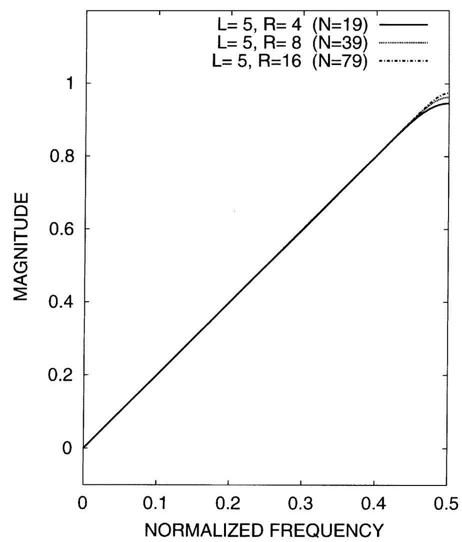

Fig. 3. Magnitude responses of Type 4 differentiators.

near the point $\omega = 3\pi/4$. As the regularity R increases, the maximum deviation decreases.

B. *Example 2 (Type 3 Bandpass Filters)*

Type 3 maximally flat bandpass filters of $L = 5$, whose center of the passband is 0.4π, are designed for $R = 4$, 8, and 16, by using (34) under the specification

$$a_{r,k} = A^{(r)}\left(\frac{2\pi k}{5}\right) = \begin{cases} 1, & (r = 0,\ k = 1) \\ -1, & (r = 0,\ k = 4) \\ 0, & \text{(otherwise)} \end{cases}$$

and their magnitude responces are shown in Fig. 2. They are Nyquist filters as well. The filter degrees by (29) for $R = 4$, 8, and 16 are 20, 40, and 80, respectively. However, by the Nyquist property, we get $h[0] = h[N] = 0$, which reduces the actual filter degrees by two. The filter coefficients for $R = 8$ are shown in Table III, in which we see that $h[5i] = 0$ ($i = 0, \ldots, 8$).

C. *Example 3 (Type 4 Digital Differentiators)*

Type 4 digital differentiators are designed by the design procedure in Section IV, for $L = 5$ and $R = 4$, 8, and 16. The specification matrices $(a_{r,k})$ are described as

$$a_{r,k} = A^{(r)}\left(\frac{2\pi k}{5}\right) = \begin{cases} 1 - \frac{2}{5}\left|k - \frac{5}{2}\right|, & (r = 0,\ 0 \le k \le 4) \\ \frac{1}{\pi}, & (r = 1,\ 0 \le k \le 2) \\ -\frac{1}{\pi}, & (r = 1,\ 3 \le k \le 4) \\ 0, & (r \ge 2). \end{cases}$$

The magnitude responses of the obtained differentiators are shown in Fig. 3.

VII. CONCLUDING REMARKS

In this paper, we have proposed a new method for designing linear-phase FIR filters by using the blockwise waveform moments. All four types of linear-phase FIR filters are covered.

The proposed design method is based on the relationship between the blockwise waveform moments and the derivatives of the magnitude response. This enables us to control the first R derivatives of the magnitude response at L-division frequency points. The degree of the resulting FIR filters is about LR. The design procedure consists of two stages, involving R times of L-point IDFT and L times of a linear system of size R by R. The inversion of the linear systems can be precomputed. In other words, the filter coefficients can be precomputed for the prototype filters, and others are obtained by superposition of these prototype filters. Therefore, the design procedure is computationally efficient. In addition, a closed-form formula for the maximally flat R-regular Lth-band FIR filters have been derived as well. The formula is applicable for both even and odd R and includes the formulae presented in [9].

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