Closed-Form Design of Generalized Maxflat R-Regular FIR Mth-Band Filters Using Waveform Moments

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***Abstract*—Mth-band filters have found numerous applications in multirate signal processing systems, filter banks, and wavelets. In this paper, the design problem of generalized maxflat R-regular finite impulse response (FIR) Mth-band filters with a specified integer group delay at $\omega = 0$ is considered, and the closed-form expression for its impulse response is presented. The filter coefficients are directly derived by solving a linear system of Vandermonde equations that are obtained from the regularity condition of the maxflat R-regular FIR Mth-band filters via the blockwise waveform moments. Differing from the conventional FIR Mth-band filters with exactly linear phase responses, the generalized FIR Mth-band filters proposed in this paper have an arbitrarily specified integer group delay at $\omega = 0$. Moreover, a new efficient implementation of the generalized maxflat R-regular FIR Mth-band filters is proposed by making use of the relationship between the filter coefficients in the closed-form solution. Finally, several design examples are presented to demonstrate the effectiveness of the proposed FIR Mth-band filters.**

***Index Terms*—Closed-form design, finite impulse response (FIR) filter, maxflat filter, Mth-band filter, waveform moment.**

I. Introduction

Mth-band filters are an important class of digital filters and have found numerous applications in multirate digital signal processing systems, filter banks and wavelets, and so on [1]–[4]. Its impulse response is required to be exactly zero-crossing at the Nyquist rate, except for one point. Until now, finite impulse response (FIR) Mth-band filters with exactly linear phase responses have been exhaustively studied [6]–[9], [12]–[14]. Among those works, various methods for designing the linear phase FIR Mth-band filters with the minimax (equiripple) magnitude responses have been proposed in [6]–[9], [13], and [14]. In recent years, the maxflat (maximally flat) R-regular FIR Mth-band filters play an important role in constructing regular M-band wavelet bases [12]. The closed-form solution for the maxflat R-regular FIR Mth-band filters with exactly linear phase responses has been given in [12] and [23]. However, a larger delay results when higher order FIR linear phase filters are needed. This is because the group delay is equal to half the filter order for the exactly linear phase FIR filters. A lower delay is generally needed in some applications of real-time signal processing [15]. For this reason, the design problem of FIR Mth-band filters with an arbitrarily specified group delay needs to be considered. A few design methods of FIR Mth-band filters with an arbitrarily specified group delay have been proposed in [16] by using the Remez exchange algorithm to obtain the equiripple response in the stopband and in [18] by using the eigenfilter to obtain the least squares solution. On the other hand, the design of generalized maxflat FIR half-band filters, a special case of the maxflat R-regular FIR Mth-band filters with an arbitrarily specified group delay ($M = 2$), has been studied also, and the closed-form solution has been given in [17], [19], [21], and [22]. In addition, a closed-form solution has been given in [20] only for a specific class of the maxflat R-regular FIR Mth-band filters with some specific group delays. The design of generalized maxflat R-regular FIR Mth-band filters with a specified integer group delay at $\omega = 0$ has been also solved in [24] and as a special case in [25].

In this paper, we consider the design problem of generalized maxflat R-regular FIR Mth-band filters with arbitrarily specified integer group delays at $\omega = 0$. We review and extend our previous work in [24] and present closed-form expressions for the impulse responses. In the proposed design method, we derive a linear system of Vandermonde equations from the regularity condition of the maxflat R-regular FIR Mth-band filters via the blockwise waveform moments defined in [10], and thus easily obtain a set of filter coefficients by applying the Cramer's formula and Vandermonde's determinant. Differing from the conventional FIR Mth-band filters with exactly linear phase responses, the generalized maxflat R-regular FIR Mth-band filters proposed in this paper have an arbitrarily specified integer group delay at $\omega = 0$. Moreover, a new efficient implementation of the generalized maxflat R-regular FIR Mth-band filters is presented by making use of the relationship between the filter coefficients in the closed-form solution. Finally, several examples are shown to demonstrate the effectiveness of the generalized maxflat R-regular FIR Mth-band filters.

This paper is organized as follows. The property of FIR Mth-band filters is first investigated in Section II. The definition of waveform moments and their relationship with the derivatives of the frequency response is briefly reviewed in

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Section III. In Section IV, the closed-form design of generalized maxflat R-regular FIR Mth-band filters and its efficient implementation are proposed. In Section V, the relationship of the generalized maxflat R-regular FIR Mth-band filters with the conventional linear phase FIR Mth-band filters is examined. In Section VI, several design examples are presented to demonstrate the effectiveness of the proposed FIR Mth-band filters. Finally, Section VII contains a conclusion.

II. FIR MTH-BAND FILTERS

Let the transfer function $H(z)$ of an FIR digital filter of length N be

$$H(z) = \sum_{n=0}^{N-1} h_n z^{-n} \tag{1}$$

where h_n are real filter coefficients. When $H(z)$ is designed as an Mth-band filter, its impulse response is required to be exactly zero-crossing except for one point K, i.e.,

$$h_{K+kM} = \begin{cases} \frac{1}{M}, & (k=0) \\ 0, & (k=\pm1,\pm2,\dots) \end{cases} \tag{2}$$

where K and M are integers and K corresponds to the desired group delay in the passband. In the case of FIR filters with exactly linear phase responses, the filter coefficients have to be symmetric, i.e., $h_n = h_{N-1-n}$, and then, its group delay is equal to $K=(N-1)/2$. Note that N must be an odd number. Hence, K increases with an increasing filter length N. It results in a larger delay when higher order FIR filters are needed. In some applications of real-time signal processing, a lower delay is generally required. In this paper, we will consider the design problem of FIR Mth-band filters with an arbitrarily specified integer group delay K.

Mth-band filter is required to be low-pass, and the desired frequency response is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-jK\omega}, & \text{(in passband)} \\ 0, & \text{(in stopband)} \end{cases}. \tag{3}$$

Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$, i.e., $\hat{h}_n = h_{n+K}$ for $n=-K,-K+1,\dots,N-K-1$. The desired frequency response of $\hat{H}(z)$ becomes

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1, & \text{(in passband)} \\ 0, & \text{(in stopband)} \end{cases}. \tag{4}$$

By substituting the time-domain condition of (2) into (1), we have

$$\hat{H}(z) = z^K H(z) = \frac{1}{M} + \sum_{\substack{n=-K \\ \neq kM}}^{N-K-1} \hat{h}_n z^{-n}. \tag{5}$$

It can be seen from (5) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{k=0}^{M-1} \hat{H}(e^{j(\omega+2k\pi/M)}) \equiv 1 \tag{6}$$

which means that the sum of the responses at the frequency points $\omega_k = \omega + 2k\pi/M$ for $k=0,1,\dots,M-1$ keep constant, regardless of what the filter coefficients h_n are. From (6), we get

$$\hat{H}(e^{j\omega_0}) = 1 - \sum_{k=1}^{M-1} \hat{H}(e^{j\omega_k}). \tag{7}$$

It is clear that the frequency response at ω_0 is dependent on the frequency responses at ω_k $(k=1,2,\dots,M-1)$. If its stopband response is zero, then the frequency response of $\hat{H}(z)$ will be one in the passband, i.e., the magnitude response of $H(z)$ is one, and the group delay is K in the passband [16]. Therefore, the design problem of FIR Mth-band filters with an arbitrarily specified integer group delay K can be reduced to the minimization of the error of $\hat{H}(z)$ in the stopband.

III. DEFINITION OF WAVEFORM MOMENTS

Like their well-known statistical counterparts, waveform moments describe some of the important geometric characteristics of a signal, e.g., the position of center, symmetry, etc. They have also been used to describe the characteristics of the impulse response of FIR digital filters [5]. The waveform moments have been extended to the blockwise waveform moments in [10]. The extension enables us to grasp the periodical characteristics of the impulse response. Specifically, it has been shown in [10] that the blockwise waveform moments exactly describe certain derivative behaviors (the value, tangency, curvature, $\cdots$) of the frequency response at the equally spaced frequency points and has been used to evaluate the quality of signals recovered by the interpolation filters [11]. This property has suggested to us a general design method of linear phase FIR filters such that the derivative behaviors of the magnitude response can be controlled arbitrarily [23]. In the following, we will apply the design method proposed in [23] to the design of generalized maxflat R-regular FIR Mth-band filters with an arbitrarily specified integer group delay at $\omega=0$.

For a nonnegative integer r, the rth waveform moment around zero for $\hat{h}_n$ is defined by

$$m_r = \sum_{n=-K}^{N-K-1} n^r \hat{h}_n = \sum_{n=0}^{N-1} (n-K)^r h_n \tag{8}$$

which is equivalent to the waveform moment around K for the impulse response h_n. Therefore, this definition is a generalization of the conventional waveform moment proposed in [5]. In [5] and [23], only the waveform moments around zero and around the center have been defined. This generalization enables us to design FIR filters with an arbitrarily specified K. It is known in [5] and [23] that the rth waveform moment m_r describes the rth differential coefficient of the frequency response $\hat{H}(e^{j\omega})$ at $\omega=0$

$$m_r = j^r \frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=0}. \tag{9}$$

Similarly, the blockwise waveform moment around zero for $\hat{h}_n$ is defined by

$$m_r(i) = \sum_{k=N_l(i)}^{N_u(i)} (kM+i)^r \hat{h}_{kM+i} \tag{10}$$

where $0 \leq i \leq M-1$ and

$$\begin{cases} N_l(i) = -\left\lfloor \frac{K+i}{M} \right\rfloor \\ N_u(i) = \left\lfloor \frac{N-K-i-1}{M} \right\rfloor \end{cases}.$$

Note that $\lfloor x \rfloor$ is the largest integer not greater than x. The blockwise waveform moments defined in (10) enable us to describe a periodic property of the impulse response $\hat{h}_n$. It is seen that the waveform moment m_r in (8) can be decomposed into the sum of M blockwise waveform moments $m_r(i)$;

$$m_r = \sum_{i=0}^{M-1} \sum_{k=N_l(i)}^{N_u(i)} (kM+i)^r \hat{h}_{kM+i} = \sum_{i=0}^{M-1} m_r(i). \tag{11}$$

It follows from the definition of $m_r(i)$ in (10) that

$$\left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=2k\pi/M} = (-j)^r \sum_{i=0}^{M-1} m_r(i) e^{-j(2ik\pi/M)} \tag{12}$$

i.e., the blockwise waveform moments describe the derivative behaviors of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$ ($0 \leq k \leq M-1$). It is seen in (12) that the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at $\omega_k = 2k\pi/M$ are the M-point DFT (Discrete Fourier Transform) of the blockwise waveform moments $m_r(i)$. Thus, by the inverse transform, we have

$$m_r(i) = \frac{j^r}{M} \sum_{k=0}^{M-1} \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=2k\pi/M} e^{j(2ik\pi/M)}. \tag{13}$$

It is clear that the blockwise waveform moments $m_r(i)$ bridge between the time and frequency domains by (10) and (13). Given the rth derivatives of the frequency response $\hat{H}(e^{j\omega})$ at the frequency points $\omega_k = 2k\pi/M$, the rth blockwise waveform moments $m_r(i)$ can be calculated via the IDFT in (13). Then, the filter coefficients can be obtained by solving the linear equations in (10). Therefore, general FIR filters can be designed in the derivative-controlled manner by (10) and (13).

IV. Closed-Form Design of Generalized Maxflat R-Regular FIR Mth-Band Filters

Theory of regular M-band wavelet bases has been studied in [12] as a generalization of two-band wavelet bases, since they help to zoom in onto narrow-band high-frequency components of a signal, while simultaneously having a logarithmic decomposition of frequency channels. In the construction of regular M-band wavelet bases, regular M-band scaling filters need to be designed first. It is known in [12] that this M-band scaling filters are obtained from regular Mth-band filters. Therefore, the design of Mth-band filters satisfying the R-regularity condition needs to be considered. In [12] and [23], a closed-form solution has been given for the maxflat R-regular FIR Mth-band filters with exactly linear phase responses. In particular, the closed-form solution derived via the blockwise waveform moments in [23] includes that in [12]. For the design of R-regular FIR Mth-band filters with an arbitrarily specified integer group delay K, however, a closed-form solution is given in [20] only for a specific class of the maxflat R-regular FIR Mth-band filters with $N = MR-1$ and $K = kM-1$, where $1 \leq k \leq R-1$. Since K is restricted to several specific integers, the group delay cannot be arbitrarily specified. In [25], an explicit solution of systems for simultaneous Lagrangian upsampling and fractional-sample delaying is given, where the generalized maxflat R-regular FIR Mth-band filters are considered as a special case (Nyquist solution). In the following, we will describe how to design the generalized maxflat R-regular FIR Mth-band filters with an arbitrarily specified integer group delay K.

It is known in [12] that an Mth-band filter is said to be R-regular if it has

$$H(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})^R Q(z) \tag{14}$$

where $Q(z)$ is an FIR filter of length L. First, we consider the minimal length of the generalized maxflat R-regular FIR Mth-band filters. For FIR Mth-band filter $H(z)$, $Q(z)$ must be used to satisfy the time-domain condition of the impulse response in (2). Thus, the number of filter coefficients of $Q(z)$ should agree with the number of the constraint equations. Every block having M coefficients of the impulse response h_n contains one of the constraint equations, except the last block. For the last block having less than M coefficients, the constraint may not be contained depending on the position of K. Therefore, L must satisfy

$$\left\lfloor \frac{N}{M} \right\rfloor \leq L \leq \left\lceil \frac{N}{M} \right\rceil \tag{15}$$

where $\lceil x \rceil$ is the least integer not less than x. Since the length of $H(z)$ is $N = (M-1)R + L$, (15) becomes

$$\left\lfloor \frac{L-R}{M} \right\rfloor \leq L - R \leq \left\lceil \frac{L-R}{M} \right\rceil. \tag{16}$$

It is clear that the solution for (16) is $L = R-1, R, R+1$.

1) In the case of $L = R-1$, the minimal length of $H(z)$ is $N = MR-1$. There are R blocks, but the last block has $M-1$ coefficients. Since the number of coefficients of $Q(z)$ is $L = R-1$, the last block does not contain the constraint; thus the group delay must be $K = kM-1$ for $1 \leq k \leq R-1$. Only this case has been considered in [20].
2) In the case of $L = R$, the minimal length is $N = MR$. All R blocks have M coefficients. Therefore, K can be arbitrarily specified. When $K = kM-1$ for $1 \leq k \leq R-1$, we have $h_{MR-1} = 0$, and the filter length degrades to $N = MR-1$. This case corresponds to that of $L = R-1$, thus $L = R-1$ is only a special case of $L = R$. When $K = kM$ for $1 \leq k \leq R-1$, the filter length degrades to $N = MR-1$ due to $h_0 = 0$. This case corresponds to the case where $K = kM-1$ for $1 \leq k \leq R-1$ plus one delay unit z^{-1}.

3) In the case of $L = R+1$, the filter length is $N = MR+1$. There are $R+1$ blocks, but the last block has one coefficient only. Since the number of coefficients of $Q(z)$ is $L = R+1$, the last block must contain the constraint. Thus, the group delay is $K = kM$ for $0 \leq k \leq R$. When $K = 0$ or $K = RM$, the minimal length is $N = MR$ since $h_{MR} = 0$ or $h_0 = 0$. When $K = kM$ for $1 \leq k \leq R-1$, due to $h_0 = h_{MR} = 0$, the minimal length degrades to $N = MR-1$. Therefore, the case of $L = R+1$ is included in that of $L = R$.

To summarize the above discussion, the minimal length of the generalized maxflat R-regular FIR Mth-band filters is $N = MR$, except for a special case where $K = kM-1$ for $1 \leq k \leq R-1$ and then $N = MR-1$.

It is known in [12] that (14) is equivalent to

$$\left.\frac{\partial^r H(e^{j\omega})}{\partial \omega^r}\right|_{\omega=2k\pi/M} = \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=2k\pi/M} = 0 \tag{17}$$

for $k = 1, 2, \ldots, M-1$ and $r = 0, 1, \ldots, R-1$. It is obtained from (7) and (17) that

$$\begin{cases} \hat{H}(e^{j\omega})\big|_{\omega=0} = 1 & (r=0) \\ \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=0} = 0 & (r = 1, 2, \ldots, R-1) \end{cases} \tag{18}$$

which means that the magnitude response $|\hat{H}(e^{j\omega})|$ and group delay $\hat{\tau}(\omega)$ of $\hat{H}(z)$ satisfy at $\omega = 0$

$$\begin{cases} |\hat{H}(e^{j\omega})|\big|_{\omega=0} = 1 & (r=0) \\ \left.\frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\right|_{\omega=0} = 0 & (r = 1, 2, \ldots, R-1) \end{cases} \tag{19}$$

and

$$\left.\frac{\partial^r \hat{\tau}(\omega)}{\partial \omega^r}\right|_{\omega=0} = 0 \qquad (r = 0, 1, \ldots, R-2). \tag{20}$$

From the relationship between $H(z)$ and $\hat{H}(z)$ in (5), we get

$$\begin{cases} |H(e^{j\omega})|\big|_{\omega=0} = 1 & (r=0) \\ \left.\frac{\partial^r |H(e^{j\omega})|}{\partial \omega^r}\right|_{\omega=0} = 0 & (r = 1, 2, \ldots, R-1) \end{cases} \tag{21}$$

and

$$\begin{cases} \tau(\omega)\big|_{\omega=0} = K & (r=0) \\ \left.\frac{\partial^r \tau(\omega)}{\partial \omega^r}\right|_{\omega=0} = 0 & (r = 1, 2, \ldots, R-2) \end{cases}. \tag{22}$$

It is clear in (21) and (22) that $H(z)$ has flat magnitude and group delay responses at $\omega = 0$.

By using (13), the blockwise waveform moments are obtained from the regularity condition in (17) and (18) as

$$m_r(i) = \begin{cases} \frac{1}{M} & (r=0) \\ 0 & (r = 1, 2, \ldots, R-1) \end{cases}. \tag{23}$$

According to the definition of the blockwise waveform moment in (10), we derive a system of linear equations as follows:

$$\begin{cases} \sum_{k=N_l(i)}^{N_u(i)} \hat{h}_{kM+i} = \frac{1}{M} & (r=0) \\ \sum_{k=N_l(i)}^{N_u(i)} (kM+i)^r \hat{h}_{kM+i} = 0 & (r = 1, 2, \ldots, R-1) \end{cases}. \tag{24}$$

Since $\hat{h}_n = h_{n+K}$ and the minimal length is $N = MR$, we have

$$\begin{cases} \sum_{k=0}^{R-1} h_{kM+i} = \frac{1}{M} & (r=0) \\ \sum_{k=0}^{R-1} (kM+i-K)^r h_{kM+i} = 0 & (r = 1, 2, \ldots, R-1) \end{cases} \tag{25}$$

which is rewritten in matrix form as

$$\mathbf{V}\mathbf{h} = \mathbf{u} \tag{26}$$

where $\mathbf{h} = [h_i, h_{M+i}, \ldots, h_{(R-1)M+i}]^T$, $\mathbf{u} = [1/M, 0, \ldots, 0]^T$, and we have the equation shown at the bottom of the page. It should be noted that $\mathbf{V}$ is the Vandermonde matrix with distinct elements. Therefore, there is always a unique solution. By using Cramer's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$h_{kM+i} = \frac{(-1)^k \prod_{\substack{n=0 \\ \neq k}}^{R-1} (nM-K+i)}{M^R k!(R-k-1)!}. \tag{27}$$

Once M, R, and K are given, a set of filter coefficients can be easily calculated by using (27). It is seen that the resulting impulse response satisfies the time-domain condition in (2). When $M = 2$, a class of generalized maxflat FIR half-band filters proposed in [17], [19], [21], and [22] can be obtained from (27).

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ (i-K) & (M+i-K) & \cdots & ((R-1)M+i-K) \\ \vdots & \vdots & \ddots & \vdots \\ (i-K)^{R-1} & (M+i-K)^{R-1} & \cdots & ((R-1)M+i-K)^{R-1} \end{bmatrix}$$

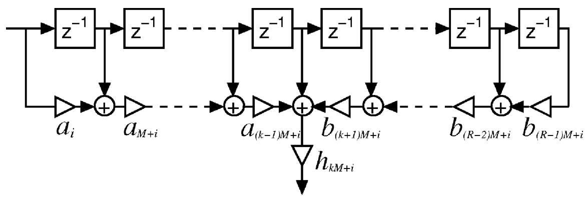


Fig. 1. A new efficient structure for the polyphase component $H_i(z)$ of the generalized maxflat R-regular FIR Mth-band filters.

The following relationship holds between the filter coefficients in (27):

$$a_{kM+i}=\frac{h_{kM+i}}{h_{(k+1)M+i}}=\frac{((k+1)M-K+i)(k+1)}{(kM-K+i)(k+1-R)} \quad (28)$$

$$b_{kM+i}=\frac{h_{kM+i}}{h_{(k-1)M+i}}=\frac{((k-1)M-K+i)(k-R)}{(kM-K+i)k}. \quad (29)$$

Note that $a_{kM+i}b_{(k+1)M+i} = 1$. It is found that the filter coefficients can be efficiently calculated by using (28) and (29) instead of (27). Furthermore, (28) and (29) will lead to a new efficient implementation of the proposed FIR Mth-band filters. We consider the polyphase representation $H(z) = \sum_{i=0}^{M-1} z^{-i}H_i(z^M)$. Except z^{-K}/M, the polyphase component $H_i(z)$ can be efficiently implemented as shown in Fig. 1. The main advantage of the new structure is that the dynamic range of the multiplication coefficients is greatly reduced. For example, by using (27), the filter coefficients of the third polyphase component $H_3(z)$ of the maxflat R-regular FIR Mth-band filters with $M = 5$, $R = 5$, $K = 12$ are

$$\left\{h_3 = -\frac{11}{3125}, h_8 = \frac{99}{3125}, h_{13} = \frac{594}{3125},\right.$$
$$\left. h_{18} = -\frac{66}{3125}, h_{23} = \frac{9}{3125}\right\}.$$

If the filter is implemented in the direct form, the ratio of the largest amplitude of the multiplication coefficients to the least one is $(594/3125)/(9/3125) = 66$. However, the multiplication coefficients in the proposed structure are

$$\left\{a_3 = -\frac{1}{9}, a_8 = \frac{1}{6}, h_{13} = \frac{594}{3125}, b_{18} = -\frac{1}{9}, b_{23} = -\frac{3}{22}\right\}$$

and then the ratio is $(594/3125)/(1/9) \approx 1.71$.

In [25], a complete and explicit solution of systems for simultaneous Lagrangian upsampling and fractional-sample delaying has been provided. The generalized maxflat R-regular FIR Mth-band filters proposed in this paper are a special case of the three-parameter family of systems proposed in [25] when the group delay is integer (Nyquist solution). When the delay μ is not integer, i.e., $\mu \neq K$, the three-parameter family of systems does not satisfy the time-domain condition of (2). If the time-domain condition is relaxed, we can take advantage of the generalization of the blockwise waveform moment around μ and get the same solution as in (27) just by replacing K with μ, since the frequency response of the three-parameter family of systems meets (17) and (18). The four-parameter family of systems has been also discussed in [25]. However, the design method proposed in this paper is applicable to the design of infinite impulse response (IIR) Mth-band filters, since the waveform moment can be extended to IIR filters [10].

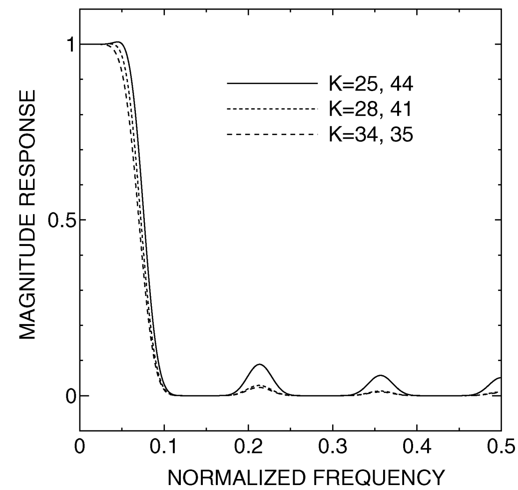


Fig. 2. Magnitude responses of the maxflat R-regular FIR Mth-band filters in Example 1.

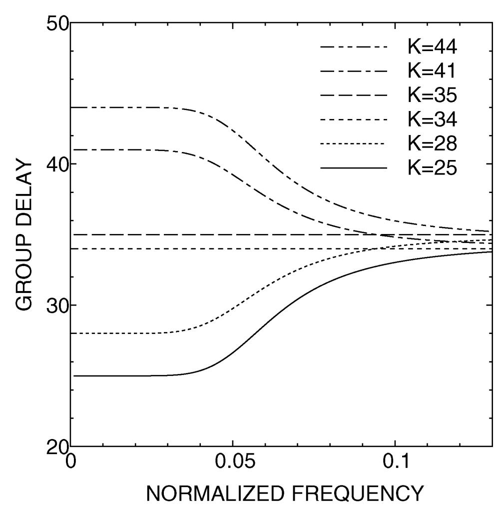


Fig. 3. Group delay responses of the maxflat R-regular FIR Mth-band filters in Example 1.

V. Relationship With the Existing Linear Phase FIR Mth-Band Filters

In this section, we examine the relationship between the generalized maxflat R-regular FIR Mth-band filters and the existing maxflat R-regular FIR Mth-band filters with exactly linear phase responses. It is well known that the filter coefficients of Type I linear phase FIR filters are symmetric, i.e., $h_n = h_{N-1-n}$. Thus, its group delay is equal to $K = (N-1)/2$, where N is an odd number. It is known in [23] that for the maxflat R-regular FIR Mth-band filters with exactly linear phase responses, the filter length N takes three values $MR-1$, MR, and $MR+1$, depending on the parity of M and R. When R is even, the minimal length is $N = MR-1$.

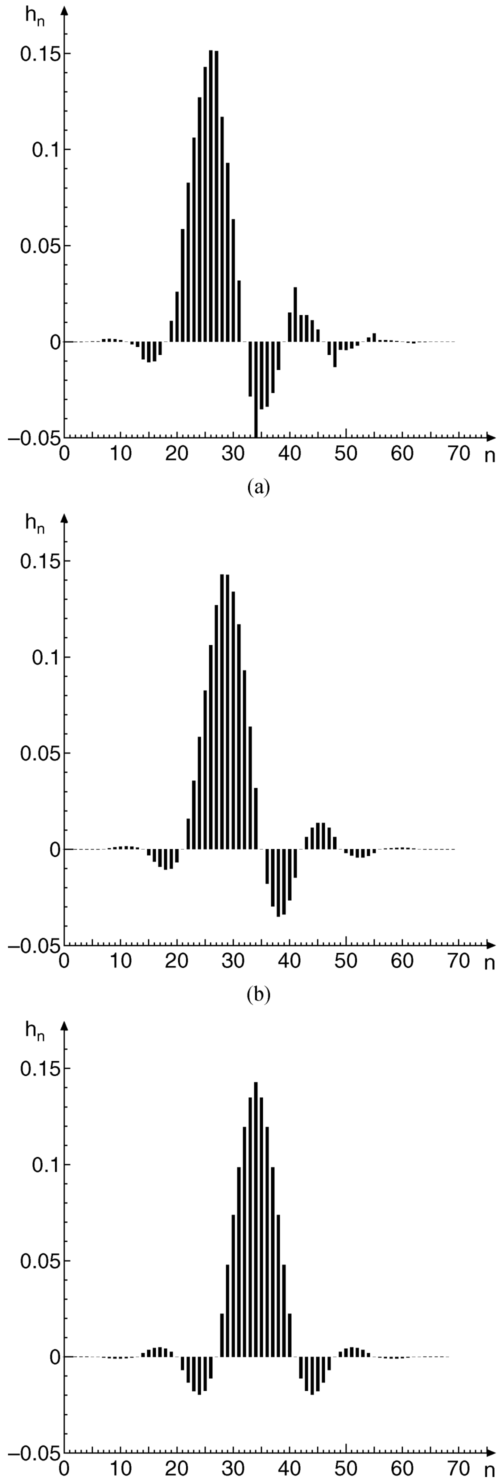


Fig. 4. Impulse responses of the maxflat R-regular FIR Mth-band filters in Example 1. (a) $K = 25$, (b) $K = 28$, and (c) $K = 34$.

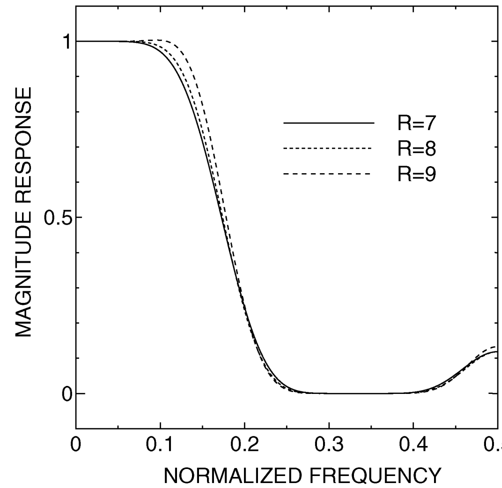


Fig. 5. Magnitude responses of the maxflat R-regular FIR Mth-band filters in Example 2.

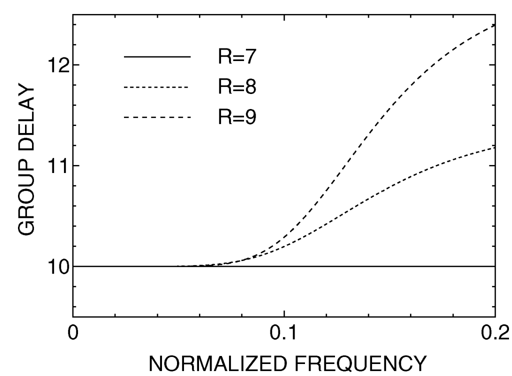


Fig. 6. Group delay responses of the maxflat R-regular FIR Mth-band filters in Example 2.

We can take $K = MR/2-1$ and then obtain the same solution as in [23] from (27). When R is odd, $N = MR$ if M is odd, while $N = MR+1$ if M is even. In the case where R and M are odd, due to $N = MR$, the solution in (27) becomes the same as in [23] by taking $K = (MR-1)/2$. In the case where R is odd and M is even, however, we cannot get the same solution. This is because the filter length is $N = MR+1$ in [23], while $N = MR$ in this paper. The filter obtained from (27) possesses only R zeros at $\omega = \pi$ to meet the regularity condition. Since R is an odd number, Type I linear phase FIR filters cannot be realized. To get an exactly linear phase response, an even number of zeros must be located at $\omega = \pi$, thus another zero at $\omega = \pi$ will be needed. We have also found that if we take $K = MR/2$, the filter coefficients obtained from (27) in this paper and [23, (34)] are the same, except a set of coefficients h_{kM}. In [23], the coefficients h_{kM} are symmetric, i.e., $h_{kM} = h_{(R-k)M}$ for $0 \le k \le R$, while it is not so for h_{kM} $(0 \le k \le R-1)$ from (27). To satisfy the symmetry of h_{kM}, we need to add another coefficient h_{RM}. By using the blockwise waveform moments, we can derive the similar equation in (26), where $\mathbf{h} = [h_0, h_M, \ldots, h_{RM}]^T$, $\mathbf{u} = [1/M, 0, \ldots, 0]^T$, and

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -\frac{RM}{2} & \frac{(2-R)M}{2} & \cdots & \frac{RM}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \left(-\frac{RM}{2}\right)^{R-1} & \left(\frac{(2-R)M}{2}\right)^{R-1} & \cdots & \left(\frac{RM}{2}\right)^{R-1} \end{bmatrix}.$$

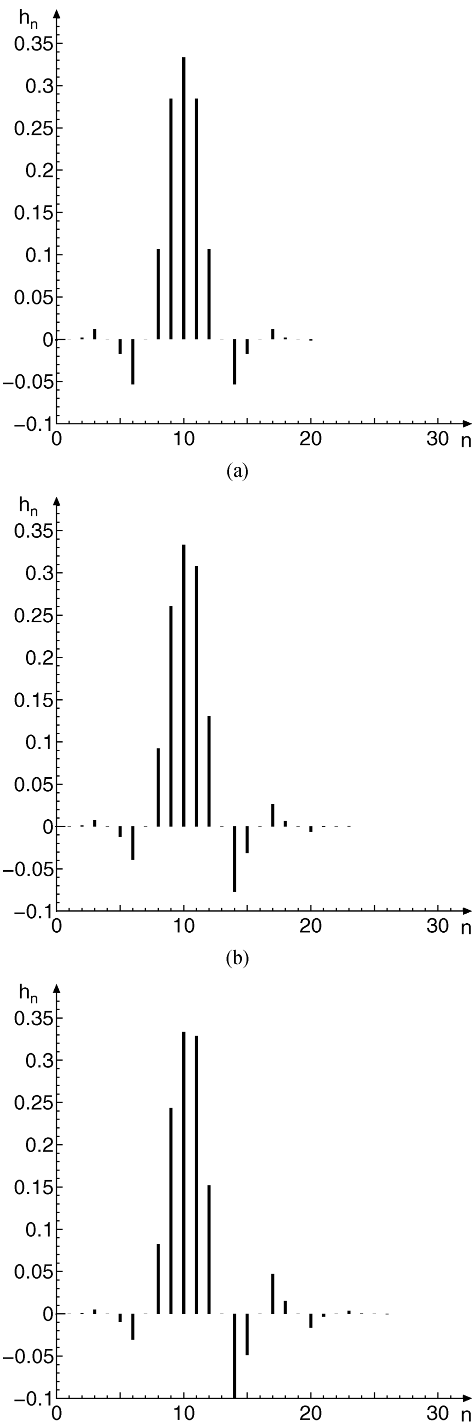


Fig. 7. Impulse responses of the maxflat R-regular FIR Mth-band filters in Example 2. (a) $R = 7$, (b) $R = 8$, and (c) $R = 9$.

Note that the matrix $\mathbf{V}$ is of size $R \times (R+1)$. By imposing the symmetry condition on h_{kM} and removing redundancies from $\mathbf{V}$, we get the closed-form solution given by

$$h_{kM} = \frac{(-1)^{k+(R+1)/2} \prod_{n=0}^{(R-1)/2} \left(n - \frac{R}{2}\right)^2}{\left(k - \frac{R}{2}\right) Mk!(R-k)!} \tag{30}$$

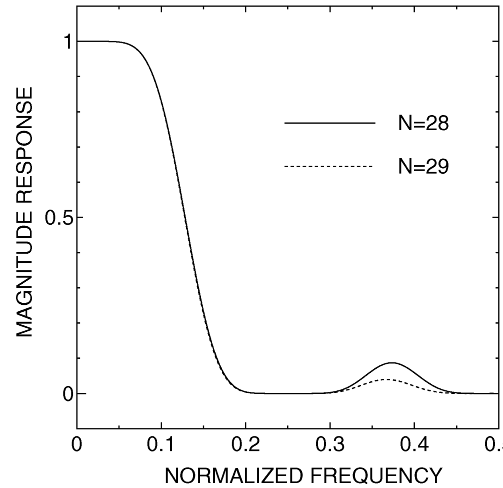


Fig. 8. Magnitude responses of the maxflat R-regular FIR Mth-band filters in Example 3.

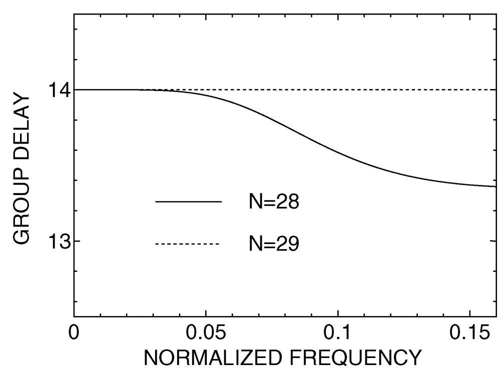


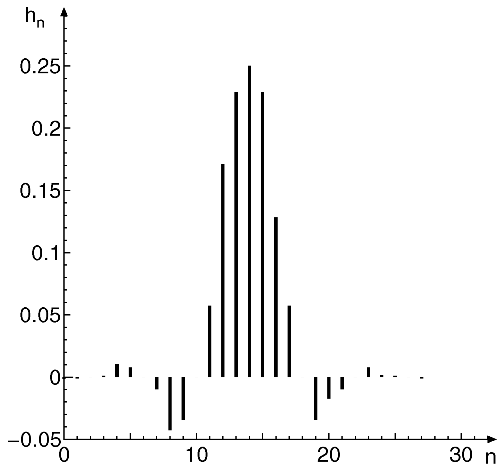
Fig. 9. Group delay responses of the maxflat R-regular FIR Mth-band filters in Example 3.

for $0 \le k \le R$. Therefore, we can obtain the same solution as in [23] from (27) by choosing an appropriate K, except for h_{kM} when R is odd and M is even, which can be calculated by (30).

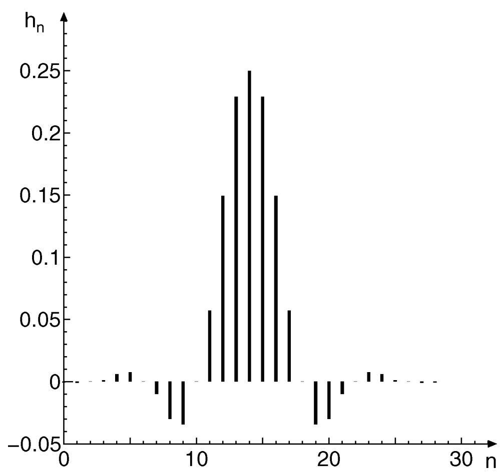
VI. Design Examples

Many design examples of the generalized maxflat FIR half-band filters, which are a special case of the generalized maxflat R-regular FIR Mth-band filters when $M = 2$, have been given in [17], [19], and [21]. In this section, we present several numerical examples to demonstrate the effectiveness of the generalized maxflat R-regular FIR Mth-band filters with $M \ge 3$ and compare the filter performance with the existing maxflat R-regular FIR Mth-band filters with exactly linear phase responses.

1) Example 1: We consider the design of the generalized maxflat R-regular FIR Mth-band filters with $M = 7$ and $R = 10$. The filter length is $N = 70$. We first take $K = 25$ and get a set of filter coefficients by (27). The resulting magnitude and group delay responses are shown in the solid line in Figs. 2 and 3, respectively, and its impulse response is shown in Fig. 4(a). We have also designed five other filters with $K = 28, 34, 35, 41, 44$. Their magnitude and group delay responses are also shown in Figs. 2 and 3. The impulse responses with $K = 28$ and $K = 34$ are shown in Fig. 4(b) and (c), respectively. When $K = 35, 41$, and 44, their impulse responses are the time-reversed versions of those with $K = 34, 28$, and 25, so they are omitted here. It is then seen in Figs. 2 and 3



(a)



(b)

Fig. 10. Impulse responses of the maxflat R-regular FIR Mth-band filters in Example 3. (a) $N = 28$ and (b) $N = 29$.

that those filters with $K = 35, 41, 44$ have the same magnitude responses as those with $K = 34, 28, 25$, while their group delay responses are symmetric. As shown in Fig. 3, the two filters with $K = 34$ and $K = 35$ have constant group delay responses at all frequencies, i.e., their phase responses are exactly linear. Thus the two filters have symmetric impulse responses, as shown in Fig. 4(c), which are correponding to the existing maxflat R-regular FIR Mth-band filters with exactly linear phase responses. Since $h_0 = 0$ when $K = 28, 35$ and $h_{69} = 0$ when $K = 34, 41$, it is noted that these filters have the actual filter length $N = 69$.

2) Example 2: We consider the design of the generalized maxflat R-regular FIR Mth-band filters with $M = 3$ and $K = 10$. We have taken $R = 7, 8, 9$ and designed three filters with $N = 21, 24, 27$. The resulting magnitude and group delay responses are shown in Figs. 5 and 6, respectively, while their impulse responses are shown in Fig. 7. It is seen that the magnitude response becomes more flat as R increases. When $R = 7$, the filter has a symmetric impulse response and an exactly linear phase response.

3) Example 3: We consider the design of the generalized maxflat R-regular FIR Mth-band filter with $M = 4$, $R = 7$, and $K = 14$. The filter length is $N = 28$. The resulting magnitude and group delay responses are shown in the solid line in Figs. 8 and 9, respectively, while its impulse response is shown in Fig. 10(a). It is seen in Fig. 10(a) that the filter coefficients are symmetric except h_{4k}. To get the exactly linear phase response, we add another coefficient h_{28}; then the filter length becomes $N = 29$. By using (30), we calculate h_{4k} to get the linear phase filter, whose magnitude, group delay, and impulse responses are shown in Figs. 8–10(b), respectively.

VII. Conclusion

In this paper, the design problem of the generalized maxflat R-regular FIR Mth-band filters with an arbitrarily specified integer group delay at $\omega = 0$ has been considered. A new closed-form expression for its impulse response has been presented. In the proposed design method, the filter coefficients are directly derived by solving a linear system of Vandermonde equations that are obtained from the regularity condition of the maxflat R-regular FIR Mth-band filters via the blockwise waveform moments. Differing from the conventional FIR Mth-band filters with exactly linear phase responses, the proposed maxflat R-regular FIR Mth-band filters have an arbitrarily specified integer group delay at $\omega = 0$. Moreover, a new efficient implementation for the generalized maxflat R-regular FIR Mth-band filters has been presented. Finally, several examples have been designed to demonstrate the effectiveness of the generalized maxflat R-regular FIR Mth-band filters.

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