Maxflat Fractional Delay IIR Filter Design

Xi Zhang, *Senior Member, IEEE*

***Abstract*—Fractional delay (FD) filters are an important class of digital filters and are useful in various signal processing applications. This paper discusses a design problem of FD infinite-impulse-response (IIR) filters with the maxflat frequency response in frequency domain. First, a flatness condition of FD filters at an arbitrarily specified frequency point is described, and then a system of linear equations is derived from the flatness condition. Therefore, a set of filter coefficients can be easily obtained by solving this system of linear equations. For a special case in which the frequency response is required to be maxflat at $\omega = 0$ or π, a closed-form expression for its filter coefficients is derived by solving a linear system of Vandermonde equations. It is also shown that the existing maxflat FD finite-impulse-response (FIR) and IIR filters are special cases of the FD IIR filters proposed in this paper. Finally, some examples are presented to demonstrate the effectiveness of the proposed filters.**

***Index Terms*—Degree of flatness, fractional delay, infinite-impulse-response (IIR) digital filter, maxflat filter design.**

I. Introduction

FRACTIONAL DELAY (FD) filters are an important class of digital filters, and have been found numerous applications in signal processing, image processing, and so on [2], [3]. FD filters are required to have a specified fractional delay and constant magnitude response. Thus, both the magnitude and group delay (or phase) responses need to be approximated simultaneously. The least-square, minimax (Chebyshev), and maxflat (maximally flat) criteria can be used for this simultaneous approximation of FD filters, as described in [2]. In this paper, we will use the maxflat criterion to optimize the frequency response of FD filters. Conventionally, finite-impulse-response (FIR) filters have been used in the design of FD filters [2]–[7], [9], [12]. It has been shown in [4]–[6] that the maxflat approximation leads to the closed-form solution of FD FIR filters with the maxflat frequency response at $\omega = 0$. Compared with FIR filters, infinite-impulse-response (IIR) filters except allpass filters are seldom used to design FD filters. Allpass IIR filters have a constant magnitude response at all frequencies, thus are suitable for using as FD filters. For allpass filter design, only its group delay or phase response needs to be optimized. The closed-form solution has been given in [8] for allpass FD filters with the maxflat frequency response at $\omega = 0$, while the equiripple phase approximation for allpass filters was also proposed in [10]. In recent years, a general class of maxflat FD IIR filters have been discussed in [14]. In [14], the Peano kernel is used in time domain to derive the closed-form solution of FD IIR filters with the maximal degree of exactness, which correspond to a class of FD IIR filters with the maxflat frequency response at $\omega = 0$.

In this paper, we discuss the design problem of FD IIR filters with the maxflat frequency response at arbitrarily specified frequency points in frequency domain. Firstly, we describe a flatness condition of FD filters at the specified frequency point. Then, we derive a system of linear equations from the flatness conditions in frequency domain, and thus can obtain easily a set of filter coefficients by directly solving this system of linear equations. We also consider a special case in which the frequency response is required to be maxflat at $\omega = 0$ or π. In this case, it is seen that the linear equations become the Vandermonde ones, thus, we can solve the linear system of Vandermonde equations to get a closed-form expression for its filter coefficients. It is also shown that the maxflat FD IIR filters proposed in this paper include the existing maxflat FD FIR and IIR filters as special cases. Finally, some design examples are presented to demonstrate the effectiveness of the maxflat FD IIR filters, and to investigate the causality and stability of the proposed IIR filters.

II. FD IIR Filters

Let $H(z)$ be the transfer function of a general class of IIR digital filters

$$H(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}} \tag{1}$$

where $N(\in Z)$ and $M(\in Z)$ are degrees of numerator and denominator, respectively, $a_n, b_m (\in R)$ are real filter coefficients, and $b_0 = 1$. If $M = 0$, then $H(z)$ in (1) becomes a FIR filter. Also if $N = M$ and $a_n = b_{N-n}$, then it will be an allpass filter.

The frequency response of $H(z)$ is given by

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)} = \frac{\sum_{n=0}^{N} a_n e^{-jn\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}} \tag{2}$$

where $|H(e^{j\omega})|$ is the magnitude response of $H(z)$, and $\theta(\omega)$ is its phase response. Then, the group delay response of $H(z)$ is defined by

$$\tau(\omega) = -\frac{\partial \theta(\omega)}{\partial \omega}. \tag{3}$$

The desired frequency response of FD filters is given by

$$H_d(e^{j\omega}) = e^{-j\{(K+s)\omega + \theta_0\}} \ (0 \le \omega \le \pi) \tag{4}$$

Manuscript received July 25, 2008; accepted February 24, 2009. First published March 24, 2009; current version published July 15, 2009. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Soontorn Oraintara.

The author is with the Department of Information and Communication Engineering, University of Electro-Communications, Tokyo 182-8585, Japan (e-mail: xiz@ice.uec.ac.jp).

Digital Object Identifier 10.1109/TSP.2009.2019231

where $K(\in Z)$ is an integer delay, $s(\in R)$ is a fractional delay in the range $[-0.5, 0.5]$, and $\theta_0(\in R)$ is a constant phase offset in $[-\pi, \pi]$. In the literature [2]–[9], [14], only the case of $\theta_0 = 0$ has been addressed. Here, we will discuss more genaral cases with any given θ_0.

Let $E(\omega)$ be the weighted error function between $H(e^{j\omega})$ and $H_d(e^{j\omega})$

$$E(\omega) = W(\omega)[H(e^{j\omega}) - H_d(e^{j\omega})] \tag{5}$$

where $W(\omega)$ is a real and positive weighting function. Therefore, the design problem of FD filters is the approximation of $H(e^{j\omega})$ to $H_d(e^{j\omega})$, that is, the minimization of the weighted error function $E(\omega)$ in the specified criterion, e.g., in the least-square, or minimax, or maxflat sense.

III. Maxflat FD IIR Filters

In this section, we describe the design of FD IIR filters in the maxflat sense. Both the magnitude and group-delay responses are required to be flat at the specified frequency point $\omega = \omega_p$. Thus, the flatness conditions are given by

$$\begin{cases} |H(e^{j\omega})||_{\omega=\omega_p} & = 1 \\ \left.\frac{\partial^r |H(e^{j\omega})|}{\partial \omega^r}\right|_{\omega=\omega_p} & = 0 \ (r = 1, 2, \ldots, R_1 - 1) \end{cases} \tag{6}$$

$$\begin{cases} \tau(\omega)|_{\omega=\omega_p} & = K + s \\ \left.\frac{\partial^r \tau(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} & = 0 \ (r = 1, 2, \ldots, R_2 - 2) \end{cases} \tag{7}$$

where $R_1(\in Z)$ and R_2 $(\in Z)$ are parameters that control the degree of flatness of the magnitude and group delay responses, respectively. In the following, we will discuss only the case of $R = R_1 = R_2$.

Let $\hat{H}(e^{j\omega}) = H(e^{j\omega})e^{j\{(K+s)\omega+\theta_0\}}$. From (1), we have

$$\hat{H}(e^{j\omega}) = H(e^{j\omega})e^{j\{(K+s)\omega+\theta_0\}} \tag{8}$$

$$= \frac{\sum_{n=0}^{N} a_n e^{j\{(K+s-n)\omega+\theta_0\}}}{\sum_{m=0}^{M} b_m e^{-jm\omega}} = \frac{N(\omega)}{D(\omega)} \tag{9}$$

where

$$\begin{cases} N(\omega) & = \sum_{n=0}^{N} a_n e^{j\{(K+s-n)\omega+\theta_0\}} \\ D(\omega) & = \sum_{m=0}^{M} b_m e^{-jm\omega}. \end{cases} \tag{10}$$

Then the desired frequency response of $\hat{H}(e^{j\omega})$ becomes

$$\hat{H}_d(e^{j\omega}) = 1. \tag{11}$$

It is clear from (9) that

$$\begin{cases} |\hat{H}(e^{j\omega})| = |H(e^{j\omega})| \\ \hat{\tau}(\omega) = \tau(\omega) - (K + s) \end{cases} \tag{12}$$

where $|\hat{H}(e^{j\omega})|$ and $\hat{\tau}(\omega)$ are the magnitude and group delay responses of $\hat{H}(e^{j\omega})$, respectively. Therefore, the flatness conditions in (6) and (7) are equivalent to

$$\begin{cases} |\hat{H}(e^{j\omega})||_{\omega=\omega_p} & = 1 \\ \left.\frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\right|_{\omega=\omega_p} & = 0 (r = 1, 2, \ldots, R - 1) \end{cases} \tag{13}$$

$$\left.\frac{\partial^r \hat{\tau}(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} = 0 (r = 0, 1, \ldots, R - 2). \tag{14}$$

It should be noted that $R = R_1 = R_2$.

Assume that $\hat{\theta}(\omega)$ is the phase response of $\hat{H}(e^{j\omega})$, then we have

$$\hat{\tau}(\omega) = -\frac{\partial \hat{\theta}(\omega)}{\partial \omega}. \tag{15}$$

It is also required that the actual phase of the filter is equal to the desired phase at $\omega = \omega_p$, i.e., $\hat{\theta}(\omega_p) = 0$. Thus, the condition in (14) becomes

$$\left.\frac{\partial^r \hat{\theta}(\omega)}{\partial \omega^r}\right|_{\omega=\omega_p} = 0 \quad (r = 0, 1, \ldots, R - 1). \tag{16}$$

Theorem 1: The flatness conditions in (13) and (16) are equivalent to

$$\begin{cases} \hat{H}(e^{j\omega})|_{\omega=\omega_p} & = 1 \\ \left.\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\right|_{\omega=\omega_p} & = 0 \ (r = 1, 2, \ldots, R - 1). \end{cases} \tag{17}$$

Proof: Since $\hat{H}(e^{j\omega}) = |\hat{H}(e^{j\omega})|e^{j\hat{\theta}(\omega)}$, then $\hat{H}(e^{j\omega_p}) = 1$ means $|\hat{H}(e^{j\omega_p})| = 1$ and $\hat{\theta}(\omega_p) = 0$, and vice versa. We have

$$\begin{aligned} \frac{\partial \hat{H}(e^{j\omega})}{\partial \omega} &= \frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} e^{j\hat{\theta}(\omega)} + |\hat{H}(e^{j\omega})| \frac{\partial e^{j\hat{\theta}(\omega)}}{\partial \omega} \\ &= \left[\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} + j|\hat{H}(e^{j\omega})| \frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right] e^{j\hat{\theta}(\omega)}. \end{aligned} \tag{18}$$

Since $|\hat{H}(e^{j\omega_p})| = 1$ and $\hat{\theta}(\omega_p) = 0$, then

$$\left.\frac{\partial \hat{H}(e^{j\omega})}{\partial \omega}\right|_{\omega=\omega_p} = \left.\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega}\right|_{\omega=\omega_p} + j \left.\frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right|_{\omega=\omega_p}. \tag{19}$$

Thus, $\partial |\hat{H}(e^{j\omega})|/\partial \omega|_{\omega=\omega_p} = 0$ and $\partial \hat{\theta}(\omega)/\partial \omega|_{\omega=\omega_p} = 0$ are equivalent to $\partial \hat{H}(e^{j\omega})/\, \partial \omega|_{\omega=\omega_p} = 0$. Similarly, we have

$$\begin{aligned} \frac{\partial^2 \hat{H}(e^{j\omega})}{\partial \omega^2} &= \Bigg[\frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2} + j \Bigg(\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} \frac{\partial \hat{\theta}(\omega)}{\partial \omega} \\ &\qquad + |\hat{H}(e^{j\omega})| \frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\Bigg)\Bigg] e^{j\hat{\theta}(\omega)} \\ &\quad + j \left[\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} + j|\hat{H}(e^{j\omega})| \frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right] \frac{\partial \hat{\theta}(\omega)}{\partial \omega} e^{j\hat{\theta}(\omega)} \\ &= \Bigg[\frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2} + 2j \frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} \frac{\partial \hat{\theta}(\omega)}{\partial \omega} \\ &\quad - |\hat{H}(e^{j\omega})| \left(\frac{\partial \hat{\theta}(\omega)}{\partial \omega}\right)^2 + j|\hat{H}(e^{j\omega})| \frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\Bigg] e^{j\hat{\theta}(\omega)}. \end{aligned} \tag{20}$$

Since $|\hat{H}(e^{j\omega_p})| = 1$, $\hat{\theta}(\omega_p) = 0$, and $\partial \hat{H}(e^{j\omega})/\, \partial \omega|_{\omega=\omega_p} = 0$, $\partial \hat{\theta}(\omega)/\partial \omega|_{\omega=\omega_p} = 0$, then

$$\left.\frac{\partial^2 \hat{H}(e^{j\omega})}{\partial \omega^2}\right|_{\omega=\omega_p} = \left.\frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2}\right|_{\omega=\omega_p} + j \left.\frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\right|_{\omega=\omega_p} \tag{21}$$

which means that $\partial^2|\hat{H}(e^{j\omega})|/\partial\omega^2|_{\omega=\omega_p} = 0$ and $\partial^2\hat{\theta}(\omega)/\partial\omega^2|_{\omega=\omega_p} = 0$ are equivalent to $\partial^2\hat{H}(e^{j\omega})/\partial\omega^2|_{\omega=\omega_p} = 0$.

When $r = 3, 4, \ldots, R-1$, $\partial^i\hat{H}(e^{j\omega})/\partial\omega^i|_{\omega=\omega_p} = 0$ and $\partial^i\hat{\theta}(\omega)/\partial\omega^i|_{\omega=\omega_p} = 0$ for $i = 1, 2, \ldots, r-1$, then we have

$$\left.\frac{\partial^r\hat{H}(e^{j\omega})}{\partial\omega^r}\right|_{\omega=\omega_p} = \left.\frac{\partial^r|\hat{H}(e^{j\omega})|}{\partial\omega^r}\right|_{\omega=\omega_p} + j\left.\frac{\partial^r\hat{\theta}(\omega)}{\partial\omega^r}\right|_{\omega=\omega_p}. \tag{22}$$

Therefore, it can be seen that $\partial^r|\hat{H}(e^{j\omega})|/\partial\omega^r|_{\omega=\omega_p} = 0$ and $\partial^r\hat{\theta}(\omega)/\partial\omega^r|_{\omega=\omega_p} = 0$ are equivalent to $\partial^r\hat{H}(e^{j\omega})/\partial\omega^r|_{\omega=\omega_p} = 0$. ■

Theorem 2: The condition in (17) is equivalent to

$$\left.\frac{\partial^r N(\omega)}{\partial\omega^r}\right|_{\omega=\omega_p} = \left.\frac{\partial^r D(\omega)}{\partial\omega^r}\right|_{\omega=\omega_p} \quad (r = 0, 1, \ldots, R-1). \tag{23}$$

Proof: From (9), we have $N(\omega) = \hat{H}(e^{j\omega})D(\omega)$, then $\hat{H}(e^{j\omega_p}) = 1$ means $N(\omega_p) = D(\omega_p)$.

We have

$$\frac{\partial N(\omega)}{\partial\omega} = \frac{\partial\hat{H}(e^{j\omega})}{\partial\omega}D(\omega) + \hat{H}(e^{j\omega})\frac{\partial D(\omega)}{\partial\omega}. \tag{24}$$

Thus, $\partial\hat{H}(e^{j\omega})/\partial\omega|_{\omega=\omega_p} = 0$ is equivalent to

$$\left.\frac{\partial N(\omega)}{\partial\omega}\right|_{\omega=\omega_p} = \left.\frac{\partial D(\omega)}{\partial\omega}\right|_{\omega=\omega_p}. \tag{25}$$

For $r = 2, 3, \ldots, R-1$, we have

$$\frac{\partial^r N(\omega)}{\partial\omega^r} = \sum_{i=0}^{r}\binom{r}{i}\frac{\partial^{r-i}\hat{H}(e^{j\omega})}{\partial\omega^{r-i}}\frac{\partial^i D(\omega)}{\partial\omega^i}. \tag{26}$$

Since $\hat{H}(e^{j\omega_p}) = 1$ and $\partial^i\hat{H}(e^{j\omega})/\partial\omega^i|_{\omega=\omega_p} = 0$ for $i = 1, \ldots, r-1$, then $\partial^r\hat{H}(e^{j\omega})/\partial\omega^r|_{\omega=\omega_p} = 0$ is equivalent to

$$\left.\frac{\partial^r N(\omega)}{\partial\omega^r}\right|_{\omega=\omega_p} = \left.\frac{\partial^r D(\omega)}{\partial\omega^r}\right|_{\omega=\omega_p}. \tag{27}$$

■

According to Theorems 1 and 2, the flatness conditions of FD filters have been reduced to (23). By substituting $N(\omega)$ and $D(\omega)$ in (10) into (23), we derive a system of linear equations as follows:

$$\sum_{n=0}^{N}(K+s-n)^r e^{j\{(K+s-n)\omega_p+\theta_0\}}a_n = \sum_{m=0}^{M}(-m)^r e^{-jm\omega_p}b_m, \tag{28}$$

for $r = 0, 1, \ldots, R-1$.

If $\omega_p \neq 0$ and $\omega_p \neq \pi$, we use $b_0 = 1$ and divide (28) into

$$\begin{cases} \sum\limits_{n=0}^{N}(K+s-n)^r\cos\{(K+s-n)\omega_p+\theta_0\}a_n \\ \qquad - \sum\limits_{m=1}^{M}(-m)^r\cos(m\omega_p)b_m = \delta(r) \\ \sum\limits_{n=0}^{N}(K+s-n)^r\sin\{(K+s-n)\omega_p+\theta_0\}a_n \\ \qquad + \sum\limits_{m=1}^{M}(-m)^r\sin(m\omega_p)b_m = 0 \end{cases}, \tag{29}$$

where

$$\delta(r) = \begin{cases} 1 & (r=0) \\ 0 & (r \neq 0). \end{cases} \tag{30}$$

Therefore, we can solve the system of linear equations in (29) to obtain a set of filter coefficients if $R = (N+M+1)/2$. It should be noted that $N+M+1$ must be an even number. The resulting FD IIR filters have the maxflat frequency response at $\omega = \omega_p$. Only computation required in the design is to solve the system of linear equations in (29).

A. A Special Case With $\omega_p = 0$

When $\omega_p = 0$, the phase offset must be $\theta_0 = 0$, or $\pm\pi$, since only the filters with the real-valued coefficients are considered in this paper. We discuss only the case of $\theta_0 = 0$, because in the case of $\theta_0 = \pm\pi$, we just need change the sign of a_n in the case of $\theta_0 = 0$. Therefore, (28) becomes

$$\sum_{n=0}^{N}(K+s-n)^r a_n - \sum_{m=1}^{M}(-m)^r b_m = \delta(r) \tag{31}$$

for $r = 0, 1, \ldots, R-1$, which is rewritten in matrix form as

$$\boldsymbol{V}\boldsymbol{a} = \boldsymbol{u} \tag{32}$$

where $\boldsymbol{a} = [a_0, a_1, \ldots, a_N, b_1, \ldots, b_M]^T$, $\boldsymbol{u} = [1, 0, \ldots, 0]^T$ [see the equation, shown at the bottom of the page]. It should be noted that $\boldsymbol{V}$ is the Vandermonde matrix with distinct elements

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ K+s & K+s-1 & \cdots & K+s-N & -(-1) & \cdots & -(-M) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (K+s)^{R-1} & (K+s-1)^{R-1} & \cdots & (K+s-N)^{R-1} & -(-1)^{R-1} & \cdots & -(-M)^{R-1} \end{bmatrix}.$$

if $s \neq 0$.[1] Therefore, there always exists a unique solution if $R = N + M + 1$. By using the Cramér's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$\begin{cases} a_n = (-1)^{n+1} \frac{M!}{n!(N-n)!} \frac{\prod_{i=0}^{N}(i-K-s)}{\prod_{i=0}^{M}(i-n+K+s)} \\ b_m = (-1)^m \frac{M!}{m!(M-m)!} \prod_{i=0}^{N} \frac{i-K-s}{i-m-K-s} \end{cases} \tag{33}$$

for $n = 0, 1, \ldots, N$ and $m = 1, 2, \ldots, M$. Once N, M, K and s are given, a set of filter coefficients a_n and b_m can be easily calculated by using (33). If we set $M = 0$ in (33), then the maxflat FD FIR filters proposed in [4]–[6] can be obtained. Also if $N = M$, then it is seen that $a_n = b_{N-n}$, thus the resulting filters become the maxflat allpass filters proposed in [8]. Therefore, it is clear that the existing maxflat FD FIR and allpass filters are two special cases of the proposed maxflat FD IIR filters. In [14], the same closed-form solution as (33) has been also derived by using the Peano kernel in time domain. The main difference is that in this paper, we have derived it from the flatness conditions in frequency domain, thus the condition can be imposed at the arbitrarily specified frequency points, whereas in [14], the derivation was done in time domain, so it is difficult to impose the flatness condition at any frequency point except $\omega_p = 0$.

B. A Special Case With $\omega_p = \Pi$

When $\omega_p = \pi$, the phase offset must satisfy $(K+s)\pi + \theta_0 = k\pi$, where $k \in Z$. It is seen that if $k = K$, then $\theta_0 = -s\pi$, and if $k = K \pm 1$, then $\theta_0 = (1-s)\pi$ for $0 < s \leq 0.5$, and $\theta_0 = -(1+s)\pi$ for $-0.5 \leq s < 0$. Here we discuss the case of even k. Similarly, just the sign of a_n need to be changed for odd k. Therefore, (28) becomes

$$\sum_{n=0}^{N}(K+s-n)^r(-1)^n a_n - \sum_{m=1}^{M}(-m)^r(-1)^m b_m = \delta(r) \tag{34}$$

for $r = 0, 1, \ldots, N+M$, and then the closed-form solution is

$$\begin{cases} a_n = -\frac{M!}{n!(N-n)!} \frac{\prod_{i=0}^{N}(i-K-s)}{\prod_{i=0}^{M}(i-n+K+s)} \\ b_m = \frac{M!}{m!(M-m)!} \prod_{i=0}^{N} \frac{i-K-s}{i-m-K-s} \end{cases} \tag{35}$$

for $n = 0, 1, \ldots, N$ and $m = 1, 2, \ldots, M$.

C. Extension to Multiple Frequency Points

An extension of FD filters with the maxflat frequency response at a single point to multiple frequency points is straightforward. Let $R_{p0}, R_{p1}, \ldots, R_{p(L-1)}, R_{pL}$ be parameters that control the degree of flatness at $\omega = 0, \omega_{p1}, \ldots, \omega_{p(L-1)}, \pi$, respectively, where $0 < \omega_{p1} < \cdots < \omega_{p(L-1)} < \pi$. If $R_{p0} + 2(R_{p1} + \cdots + R_{p(L-1)}) + R_{pL} = N + M + 1$, we can solve the system of linear equations in (29), (31), and (34) to obtain FD IIR filters with the maxflat frequency response at multiple frequency points.

[1]When $s = 0$, the desired delay is integer K, and can be easily realized by z^{-K}. So this case needs not be considered in the design.

IV. Stability Issue

In the previous section, we have described the design of the maxflat FD IIR filters. However, it is not known whether or not the obtained IIR filters are causal stable. In the following, we will discuss the causality and stability of the maxflat FD IIR filters. Since the filter coefficients can be easily obtained only by solving a system of linear equations, we can examine the causality and stability of the filters by checking whether its poles are located inside the unit circle. We have found that the proposed maxflat FD IIR filters seldom have its poles at the unit circle. However, its poles may be located outside the unit circle depending on the group delay $K+s$. IIR filters with some poles located outside the unit circle are not causal stable, but can be divided into the causal and anticausal stable parts that have the poles inside and outside the unit circle respectively. Such IIR filters are realizable in some applications such as image processing and offline processing, and have a potential of having a better frequency response [1]. When causal stable IIR filters are needed, for example, in real-time processing applications, we have to choose the group delay $K+s$ carefully. It is known in [8] and [10] that allpass filters with $N = M$ become causal stable if the group delay satisfies $K+s > N-1$. Moreover, IIR half-band filters, a special case of FD IIR filters with $s = 0.5$, have been also discussed in [11]. It has been pointed out in [11] that the filters are causal stable when the group delay is larger than a specific value, which is dependent on N and M. Similarly, we have found that the maxflat FD IIR filters with $\omega_p = 0$ or π become causal stable if we choose the desired group delay $K+s$ to be larger than a specific value, which is dependent on the filter degree N and M. However, when $\omega_p \neq 0$ and $\omega_p \neq \pi$, the situation will be more complicated, and the phase offset θ_0 and the frequency point ω_p influence the causality and stability of the filters also. See the following design examples in detail.

V. Design Examples

In this section, we present several design examples to demonstrate the effectiveness of the proposed maxflat FD IIR filters. Moreover, we investigate the causality and stability of the obtained IIR filters and compare the filter performance with the existing maxflat FD FIR and allpass filters.

Example 1: We consider the design of the maxflat FD IIR filters with $N = 8$ and $M = 4$. The degree of flatness at $\omega_p = 0$ is set to $R = N + M + 1 = 13$, and the phase offset is $\theta_0 = 0$. The integer delay is $K = 7$, and the fractional delay is chosen from $s = -0.5$ to $s = 0.5$ at intervals of $\Delta s = 0.2$. The filter coefficients are calculated by using (33), and the resulting magnitude and group delay responses are shown in Figs. 1 and 2, respectively. It is seen in Fig. 1 that the obtained magnitude responses are flat at $\omega = 0$, and are under the influence of the fractional delay s, particularly in the higher frequency. In Fig. 2, the group delay responses are also flat at $\omega = 0$, and can be

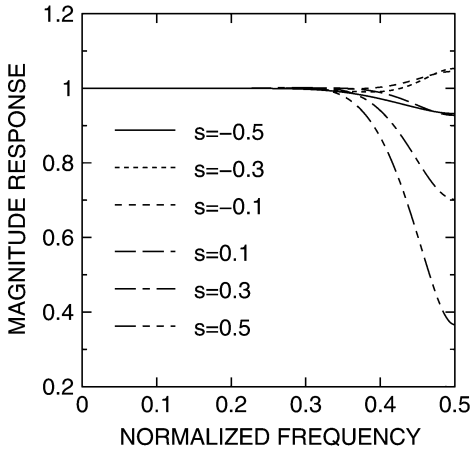

Fig. 1. Magnitude responses of the maxflat FD IIR filters in Example 1.

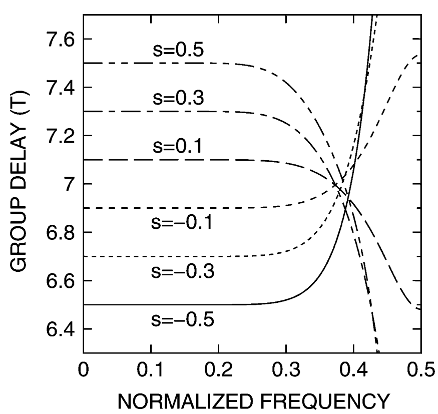

Fig. 2. Group delay responses of the maxflat FD IIR filters in Example 1.

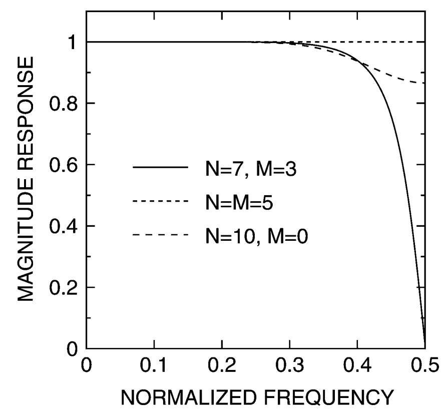

Fig. 3. Magnitude responses of the maxflat FD IIR filters in Example 2.

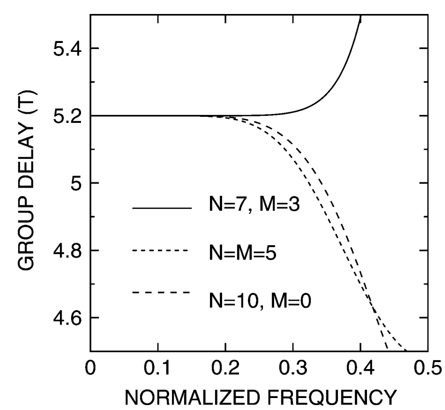

Fig. 4. Group delay responses of the maxflat FD IIR filters in Example 2.

arbitrarily specified. To examine the causality and stability of the filters, we have designed many filters by changing the group delay from $K+s=0$ to $K+s=500$ at intervals of 0.01, and then checked whether all poles are located inside the unit circle. It has been found that the obtained maxflat FD IIR filters are causal stable when $K+s>5.80$.

Example 2: We consider the design of the maxflat FD IIR filters with $N+M=10$, and the desired delay $K=5$ and $s=0.2$. The degree of flatness at $\omega_p=0$ is set to $R=N+M+1=11$, and the phase offset is $\theta_0=0$. We have designed the maxflat FD IIR filter with $N=7$ and $M=3$ by using (33). The FD IIR filters with $N=7$ and $M=3$ become causal stable when $K+s>4.64$. The resulting magnitude and group delay responses are shown in the solid line in Figs. 3 and 4, respectively. We have also designed the maxflat FD FIR filter with $N=10$ $(M=0)$, and allpass filter with $N=M=5$. Their magnitude and group delay responses are also shown in Figs. 3 and 4. It is seen in Fig. 3 that the magnitude response of allpass filter is always 1 at all frequencies, whereas the IIR filter with $N=7$ and $M=3$ has more flat group delay response than the FD FIR and allpass filters, as shown in Fig. 4.

Example 3: We consider the design of the maxflat FD IIR filters with $N=7$ and $M=4$, and the desired delay $K=6$ and $s=0.1$. The degree of flatness at $\omega_p=\pi/2$ is set to $R=(N+M+1)/2=6$. The phase offset is chosen as $\theta_0=0$ and $\theta_0=\pm 0.3\pi$. We have designed three maxflat FD IIR filters with $\theta_0=0,\pm 0.3\pi$ by solving the system of linear equations in (29). The resulting magnitude and group delay responses are shown in Figs. 5 and 6, respectively. It is seen in Figs. 5 and 6 that the magnitude and group delay responses of these filters are under the influence of the phase offset θ_0. These obtained IIR filters have all poles inside the unit circle and thus are causal stable. We have also found that when $\theta_0=0$, the filter is causal stable if $K+s>5.32$. However, when $\theta_0=-0.3\pi$, the desired group delay must satisfy $4.31<K+s<5.30$ or $K+s>5.42$ to guarantee the causality and stability, while when $\theta_0=0.3\pi$, the range having causal stable IIR filters is $4.33<K+s<4.63$, $4.70<K+s<5.71$ and $K+s>5.93$. It is clear that the causality and stability of the filters is dependent on the phase offset θ_0 also, which needs to be further investigated.

Example 4: We consider the design of the maxflat FD IIR filters with $N=9$ and $M=5$. The desired delay is $K=8$ and $s=0.3$, and the phase offset is $\theta_0=0$. The degree of flatness is $R_{p0}=5$ at $\omega_{p0}=0$, and $R_{p1}=5$ at ω_{p1}. Note that $R_{p0}+2R_{p1}=N+M+1=15$. First, we have designed the maxflat FD IIR filter with $\omega_{p1}=0.6\,\pi$. Its magnitude and group delay responses are shown in the solid line in Figs. 7 and 8, respectively. We have investigated the filters with $0<K+s<25$, and found that it is causal stable if $7.2<K+s<10.33$, $12.41<K+s<13.66$, $15.84<K+s<16.99$, $19.21<K+s<20.32$, and $22.58<K+s<23.66$. It is seen that compared with the maxflat FD IIR filters at a single frequency point, the causality and stability of the filters with multiple frequency

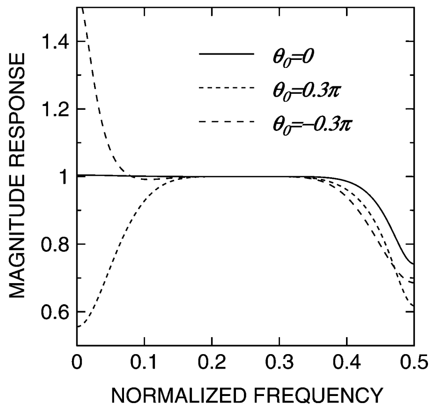

Fig. 5. Magnitude responses of the maxflat FD IIR filters in Example 3.

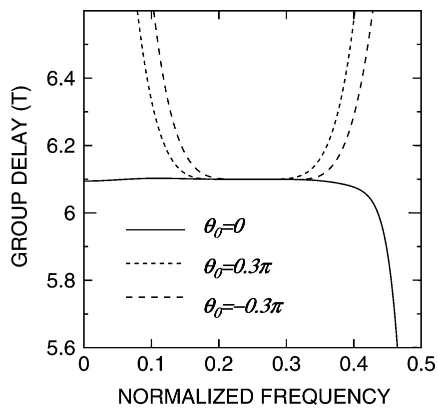

Fig. 6. Group delay responses of the maxflat FD IIR filters in Example 3.

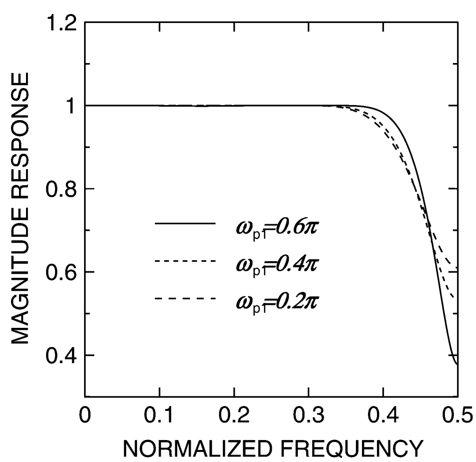

Fig. 7. Magnitude responses of the maxflat FD IIR filters in Example 4.

points is more complicated. We have designed two maxflat FD FIR filter with $\omega_{p1} = 0.4\pi$ and $\omega_{p1} = 0.2\pi$, which are also causal stable. Their magnitude and group delay responses are also shown in Figs. 7 and 8. It is found that the frequency point ω_{p1} influences the causality and stability of the filters too.

VI. Conclusion

In this paper, we have discussed the design problem of FD IIR filters with the maxflat frequency response at arbitrarily specified frequency points. A system of linear equations has been derived from the flatness conditions in frequency domain, and then

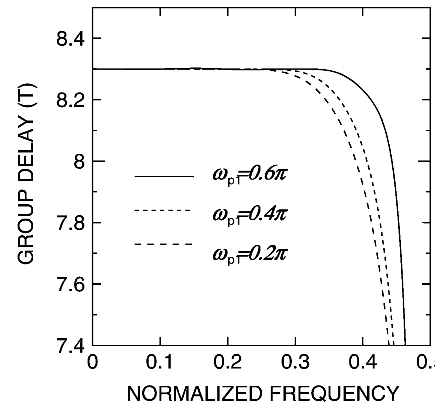

Fig. 8. Group delay responses of the maxflat FD IIR filters in Example 4.

the filter coefficients can be easily obtained by directly solving this system of linear equations. If the frequency response is required to be maxflat at $\omega = 0$ or π, a closed-form expression for its filter coefficients can be derived via the Vandermonde matrix. The maxflat FD IIR filters proposed in this paper include the existing maxflat FD FIR and IIR filters as special cases. Finally, some design examples have been shown to demonstrate the effectiveness of the maxflat FD IIR filters, and to investigate the causality and stability of the proposed filters.

References

[1] C. M. Rader, "The rise and fall of recursive digital filters," *IEEE Signal Process. Mag.*, vol. 23, no. 6, pp. 46–49, Nov. 2006.

[2] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Process. Mag.*, vol. 13, no. 1, pp. 30–60, Jan. 1996.

[3] V. Valimaki and T. I. Laakso, , F. A. Marvasti, Ed., "Fractional delay filters: Design and applications," in *Theory and Applications of Nonuniform Sampling*. New York: Kluwer, 2001.

[4] G. S. Liu and C. H. Wei, "A new variable fractional sample delay filter with nonlinear interpolation," *IEEE Trans. Circuits Syst. II*, vol. 39, no. 2, pp. 123–126, Feb. 1992.

[5] E. Hermanowicz, "Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers," *Electron. Lett.*, vol. 28, no. 20, pp. 1936–1937, Sep. 1992.

[6] P. J. Kootsookos and R. C. Williamson, "FIR approximation of fractional sample delay systems," *IEEE Trans. Circuits Syst. II*, vol. 43, no. 3, pp. 269–271, Mar. 1996.

[7] S. Samadi, A. Nishihara, and H. Iwakura, "Universal maximally flat lowpass FIR systems," *IEEE Trans. Signal Process.*, vol. 48, no. 7, pp. 1956–1964, Jul. 2000.

[8] J. P. Thiran, "Recursive digital filters with maximally flat group delay," *IEEE Trans. Circuits Theory*, vol. CT-18, no. 6, pp. 659–664, Jun. 1971.

[9] S. Samadi, M. O. Ahmad, and M. N. S. Swamy, "Results on maximally flat fractional-delay systems," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 11, pp. 2271–2286, Nov. 2004.

[10] X. Zhang and H. Iwakura, "Design of IIR digital allpass filters based on eigenvalue problem," *IEEE Trans. Signal Process.*, vol. 47, no. 2, pp. 554–559, Feb. 1999.

[11] X. Zhang and K. Amaratunga, "Closed-form design of maximally flat IIR half-band filters," *IEEE Trans. Circuits Syst. II*, vol. 49, no. 6, pp. 409–417, Jun. 2002.

[12] X. Zhang, D. Kobayashi, T. Wada, T. Yoshikawa, and Y. Takei, "Closed-form design of generalized maxflat R-regular FIR Mth-band filters using waveform moments," *IEEE Trans. Signal Process.*, vol. 54, no. 11, pp. 4214–4222, Nov. 2006.

[13] X. Zhang and D. F. Ge, "Closed-form design of maxflat R-regular IIR Mth-band filters using blockwise waveform moments," in *Proc. 15th Int. Conf. Digital Signal Processing*, Cardiff, Wales, UK, Jul. 2007.

[14] P. H. Wang, "Peano kernels of fractional delay systems," in *Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, May 2007, pp. III-1477–1480.

Xi Zhang (M'94–SM'01) received the B.E. degree in electronics engineering from the Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 1984 and the M.E. and Ph.D. degrees in communication engineering from the University of Electro-Communications (UEC), Tokyo, Japan, in 1990 and 1993, respectively.

He was with the Department of Electronics Engineering at NUAA from 1984 to 1987, and with the Department of Communications and Systems at UEC from 1993 to 1996, all as an Assistant Professor. He was with the Department of Electrical Engineering at Nagaoka University of Technology (NUT), Niigata, Japan, as an Associate Professor, from 1996 to 2004. Currently, he is with the Department of Information and Communication Engineering at UEC, as an Associate Professor. He was a Visiting Scientist of the MEXT of Japan with the Massachusetts Institute of Technology (MIT), Cambridge, from 2000 to 2001. His research interests are in the areas of digital signal processing, filter design theory, filter banks and wavelets, and its applications to image coding.

Dr. Zhang is a member of the IEICE of Japan. He received the third prize of the Science and Technology Progress Award of China in 1987, and the challenge prize of Fourth LSI IP Design Award of Japan in 2002. He served as an Associate Editor for the IEEE SIGNAL PROCESSING LETTERS from 2002 to 2004.