IIR-Based DT$\mathcal{C}$WTs With Improved Analyticity and Frequency Selectivity

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***Abstract*—In this paper, a new class of DT$\mathcal{C}$WTs with improved analyticity and frequency selectivity is proposed by using general IIR filters with numerator and denominator of different degree. In the common-factor technique proposed by Selesnick, the maximally flat allpass filter was used to satisfy the half-sample delay condition. Thus, to improve the analyticity of complex wavelets, we present a method for designing allpass filters with the specified degree of flatness and equiripple phase response in the approximation band. Furthermore, to improve the frequency selectivity of scaling lowpass filters, we locate the specified number of zeros at $z = -1$ and minimize the stopband error. The design methods proposed in this paper use the well-known Remez exchange algorithm to approximate the equiripple response. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem. Finally, we investigate the performance on the proposed DT$\mathcal{C}$WTs through several design examples. It is shown that the conventional DT$\mathcal{C}$WTs proposed by Selesnick are only the special cases of DT$\mathcal{C}$WTs proposed in this paper.**

***Index Terms*—Allpass filter, analyticity, DT$\mathcal{C}$WT, frequency selectivity, Hilbert transform pair, IIR digital filter.**

I. Introduction

THE dual tree complex wavelet transform (DT$\mathcal{C}$WT) was originally proposed by Kingsbury [5], and has been used in many applications of signal processing and image processing [6]–[10]. DT$\mathcal{C}$WTs employ two real wavelet transforms, where one wavelet corresponds to the real part of complex wavelet and the other is the imaginary part. Two wavelet bases are required to be a Hilbert transform pair. Thus, DT$\mathcal{C}$WTs are nearly shift invariant and has a good directional selectivity in two or higher dimensions with limited redundancies. It has been proved in [8], [11] and [12] that the necessary and sufficient condition for two wavelet bases to be a Hilbert transform pair is the half-sample delay condition between the corresponding scaling lowpass filters.

Several design methods for DT$\mathcal{C}$WTs have been proposed in [5]–[14] by using FIR filters, which are corresponding to the compactly supported wavelets. In [5]–[7], Kingsbury had proposed FIR Q-shift filters, whose group delay is required to be $\frac{1}{4}$ sample to satisfy the half-sample delay condition. In [9], Selesnick had proposed the common-factor technique, where the scaling lowpass filters are constructed by using allpass filters to satisfy the half-sample delay condition. This design method is simple and effective, since the approximation accuracy of the half-sample delay is controlled only by the allpass filter. Selesnick had adopted the maximally flat allpass filter and given a class of FIR orthonormal and biorthogonal solutions, and IIR orthonormal solution, where the scaling lowpass filters have as many zeros at $z = -1$ as possible to obtain the maximum number of vanishing moments of wavelets, resulting in the maximally flat magnitude responses of the scaling lowpass filters. However, the resulting IIR scaling lowpass filters have the numerator and denominator of the (almost) same degree. It is also known that the maximally flat allpass filter has a larger phase error as $|\omega|$ increases, resulting in a poor analyticity of complex wavelet. In [16], we had proposed a class of DT$\mathcal{C}$WTs using general IIR filters with improved frequency selectivity. In [17], we had also proposed a class of FIR-based DT$\mathcal{C}$WTs with improved analyticity.

In this paper, we propose a new class of DT$\mathcal{C}$WTs with improved analyticity and frequency selectivity by using general IIR filters with numerator and denominator of different degree. First of all, to improve the analyticity of complex wavelet, we present a method for designing allpass filters with the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band [17]. It is known in [2] that frequency selectivity is also a useful property for many applications of signal processing. But the maximally flat filters have a poor frequency selectivity in general. To improve the frequency selectivity of the scaling lowpass filters, we specify the number of zeros at $z = -1$ from the viewpoint of vanishing moments and then minimize the stopband error by using the remaining degree of freedom. The proposed design procedures are based on the well-known Remez exchange algorithm, thus, a set of filter coefficients can be easily obtained by solving the eigenvalue problem. The optimal solution is attained through a few iterations. It is also shown that the conventional FIR and IIR orthonormal solutions proposed in [16], [17] are only the special cases of DT$\mathcal{C}$WTs proposed in this paper. Finally, we investigate the performance on the proposed DT$\mathcal{C}$WTs and indicate how to choose the approximation band properly. The main contribution in this paper is that both the analyticity of DT$\mathcal{C}$WTs and frequency selectivity of IIR scaling lowpass filters can be improved simultaneously.

This paper is organized as follows. Section II briefly reviews DT$\mathcal{C}$WTs and the half-sample delay condition. In Section III, a class of DT$\mathcal{C}$WTs is presented by using general IIR filters. Section IV presents a design procedure for allpass filters with the specified degree of flatness to improve the analyticity.

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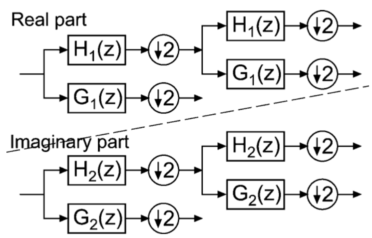

Fig. 1. Dual tree complex wavelet filter bank.

Section V gives a design procedure for scaling lowpass filters with the improved frequency selectivity. Section VI describes the performance investigation on the proposed DT$\mathcal{C}$WTs. Finally, Section VII contains a conclusion.

II. Dual Tree Complex Wavelet Transform

It is well-known in [1] that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z), G_i(z)\}$, where $i = 1, 2$. We assume that $H_i(z)$ and $G_i(z)$ are lowpass and highpass filters, respectively. The condition of orthonormality for $H_i(z)$ and $G_i(z)$ is given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0. \end{cases} \quad (1)$$

We use the notation $\phi_i(t)$, $\psi_i(t)$ to denote the scaling and wavelet functions, respectively. Thus the corresponding dilation and wavelet equations are expressed as

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n), \end{cases} \quad (2)$$

where $h_i(n)$, $g_i(n)$ are the impulse responses of $H_i(z)$, $G_i(z)$, respectively.

In [5]–[7], Kingsbury had proposed the dual tree complex wavelet transform (DT$\mathcal{C}$WT), which is constructed by two filter banks with real-coefficients, corresponding to the real and imaginary parts of the complex wavelets, that is, $\psi_c(t) = \psi_1(t) + j\psi_2(t)$, as shown in Fig. 1. Generally, two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ are required to be a pair of Hilbert transform. Thus the complex wavelet $\psi_c(t)$ is analytic, i.e., its spectrum is one-sided:

$$\Psi_c(\omega) = \Psi_1(\omega) + j\Psi_2(\omega) = \begin{cases} 2\Psi_1(\omega) & \omega > 0 \\ 0 & \omega < 0, \end{cases} \quad (3)$$

where $\Psi_i(\omega)$ is the Fourier transform of $\psi_i(t)$. However, the ideal Hilbert transform pair cannot be achieved with realizable filters. Therefore, to evaluate the analyticity, we use the p-norm of the spectrum $\Psi_c(\omega)$ to define an objective measure of quality as

$$E_p = \frac{\|\Psi_c(\omega)\|_{p,(-\infty,0)}}{\|\Psi_c(\omega)\|_{p,(0,\infty)}} \quad (4)$$

where

$$\|\Psi_c(\omega)\|_{p,\Omega} = \left(\int_\Omega |\Psi_c(\omega)|^p d\omega\right)^{\frac{1}{p}}. \quad (5)$$

If $p = \infty$, $E_\infty = \lim_{p\to\infty} E_p$ is the peak error in the negative frequency domain[13]. If $p = 2$, E_2 is the square root of the negative frequency energy. In this paper, we will use E_∞ and E_2 to evaluate the analyticity of the complex wavelet.

In [8], Selesnick had proved that two wavelet functions are a Hilbert transform pair;

$$\psi_2(t) = H\{\psi_1(t)\}. \quad (6)$$

that is,

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0), \end{cases} \quad (7)$$

if and only if corresponding scaling lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}}(|\omega| < \pi), \quad (8)$$

which is the so-called half-sample delay condition. It is the necessary and sufficient condition for two wavelet bases to be a Hilbert transform pair [11], [12]. It is seen in (8) that $H_2(e^{j\omega})$ needs to be approximated to $H_1(e^{j\omega})e^{-j\frac{\omega}{2}}$. Thus, we define the error function $E(\omega)$ to evaluate the accuracy of approximation as

$$E(\omega) = H_2(e^{j\omega}) - H_1(e^{j\omega})e^{-j\frac{\omega}{2}}. \quad (9)$$

III. The Common-Factor Technique

It is known [3] that the transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)}, \quad (10)$$

where

$$D(z) = \sum_{n=0}^{L} d(n)z^{-n}. \quad (11)$$

where L is the degree of $A(z)$ and $d(n)$ are real filter coefficients, $d(0) = 1$.

In [9], Selesnick had proposed the common factor technique where the scaling lowpass filters $H_1(z)$ and $H_2(z)$ are composed of the allpass filter by

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}). \end{cases} \quad (12)$$

Since both of scaling lowpass filters have the same factor $F(z)$, we have

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z). \quad (13)$$

It is clear that $H_2(z)$ is expressed as the product of $H_1(z)$ and $A(z)$. The half-sample delay condition in (8) can be approximately achieved if the allpass filter is an approximate half-sample delay;

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}}(|\omega| < \pi), \tag{14}$$

thus, two wavelet bases form an approximate Hilbert transform pair.

A. FIR Orthonormal Solution

After $A(z)$ is determined, $F(z)$ needs to be constructed for $H_1(z)$ and $H_2(z)$. To obtain wavelet bases with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1+z^{-1})^K. \tag{15}$$

Thus,

$$\begin{cases} H_1(z) = Q(z)(1+z^{-1})^K D(z) \\ H_2(z) = Q(z)(1+z^{-1})^K z^{-L} D(z^{-1}). \end{cases} \tag{16}$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(z+2+z^{-1})^K D(z)D(z^{-1}). \end{aligned} \tag{17}$$

Let $Q(z)$ be a FIR filter and defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-N_1}^{N_1} r(n)z^{-n}, \tag{18}$$

$$S(z) = (z+2+z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \tag{19}$$

where $r(n) = r(-n)$ for $1 \le n \le N_1$ and $s(n) = s(-n)$ for $1 \le n \le L+K$, then (17) becomes

$$P(z) = R(z)S(z). \tag{20}$$

Therefore, we can rewrite the orthonormality condition in (1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} 1 & (n=0) \\ 0 & \left(1 \le n \le \frac{N_1+L+K}{2}\right), \end{cases} \tag{21}$$

where $I_{\min} = \max\{-N_1, 2n-L-K\}$ and $I_{\max} = \min\{N_1, 2n+L+K\}$. Note that $P(z)$ is a halfband filter. The degree of $H_i(z)$ is $M = N_1 + L + K$. Since $r(n) = r(-n)$, there are totally $\frac{(M+1)}{2}$ equations with respect to (N_1+1) unknown coefficients of $r(n)$ in (21). Therefore, we can obtain the only solution if $\frac{(M+1)}{2} = N_1 + 1$. In [9], Selesnick had chosen $N_1 = L+K-1$ and obtained the filter of minimal degree for given L and K, which corresponds to the maximal K ($K_{\max} = N_1 - L + 1 = \frac{(M+1)}{2-L}$) for given L and N_1. Thus the scaling lowpass filters have the maximally magnitude responses, resulting in the maximum number of vanishing moments. This is the FIR orthonormal solution proposed in [9].

B. IIR Orthonormal Solution

In general, IIR filters require a lower computational complexity than FIR filters to achieve a sharp frequency response. IIR filters can be also used to construct $\mathrm{DT}\mathcal{C}\mathrm{WT}$. In [9], Selesnick has chosen

$$F(z) = \frac{(1+z^{-1})^K}{C(z^2)}, \tag{22}$$

then

$$\begin{cases} H_1(z) = \frac{(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \frac{(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)}. \end{cases} \tag{23}$$

$H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$,

$$P(z) = \frac{(z+2+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})}. \tag{24}$$

Defining

$$B(z) = C(z)C(z^{-1}) = \sum_{n=-N_2}^{N_2} b(n)z^{-n}, \tag{25}$$

where $b(n) = b(-n)$ for $1 \le n \le N_2$. From the orthonormality condition in (1), we have

$$S(z) + S(-z) = 2B(z^2). \tag{26}$$

thus $N_2 = \lfloor \frac{L+K}{2} \rfloor$ and

$$b(n) = s(2n), \tag{27}$$

where $\lfloor x \rfloor$ means the largest integer not greater than x. This is the IIR orthonormal solution proposed in [9]. It is clear that the numerator and denominator of $H_i(z)$ are of degree $M = L+K$ and $2N_2 = 2\lfloor \frac{L+K}{2} \rfloor$ respectively, which are almost the same.

C. General IIR Orthonormal Solution

In [16], we have proposed a new class of $\mathrm{DT}\mathcal{C}\mathrm{WTs}$ using general IIR filters with numerator and denominator of different degree. By combining FIR and IIR solutions proposed in [9], we choose

$$F(z) = \frac{Q(z)(1+z^{-1})^K}{C(z^2)}, \tag{28}$$

then

$$\begin{cases} H_1(z) = \frac{Q(z)(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \frac{Q(z)(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)}, \end{cases} \tag{29}$$

where the degree of numerator is not less than the degree of denominator, that is, $M = L+K+N_1 \ge 2N_2$. If $M > 2N_2$, then M is an odd number, whereas if $M = 2N_2$, M is an even number.

Thus, the product filter $P(z)$ is

$$P(z) = \frac{Q(z)Q(z^{-1})(1+z)^K(1+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})} = \frac{R(z)S(z)}{B(z^2)}. \quad (30)$$

From the orthonormality condition, we have

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} b(n) & (0 \le n \le N_2) \\ 0 & \left(N_2 < n \le \frac{M}{2}\right). \end{cases} \quad (31)$$

We rewrite (31) in the matrix form as

$$\mathbf{b} = \mathbf{S_1 r}, \quad (32)$$

and

$$\mathbf{S_2 r} = \mathbf{0}, \quad (33)$$

where $\mathbf{b} = [b(0), b(1), \ldots, b(N_2)]^T$, $\mathbf{r} = [r(0), r(1), \ldots, r(N_1)]^T$, $\mathbf{0} = [0, 0, \ldots, 0]^T$, and the elements of the matrices $\mathbf{S_1}$ and $\mathbf{S_2}$ are given by

$$S_1(m,n) = \begin{cases} s(2m) & (n=0) \\ s(2m-n) + s(2m+n) & (n = 1, 2, \ldots, N_1). \end{cases}$$

where $0 \le m \le N_2$, and

$$S_2(m,n) = \begin{cases} s(2(N_2+1+m)) & (n=0) \\ s(2(N_2+1+m)-n) & \\ +s(2(N_2+1+m)+n) & (n = 1, 2, \ldots, N_1). \end{cases}$$

where $0 \le m \le \lfloor \frac{M}{2} \rfloor - N_2 - 1$. Note that $s(-n) = s(n) = 0$ for $n > L + K$.

Assuming $r(0) = 1$ without any loss of generality, there are $(\lfloor \frac{M}{2} \rfloor - N_2)$ equations with respect to N_1 unknown coefficients $r(n)$ in (33). Therefore, it is clear that the only solution $r(n)$ exists if $\lfloor \frac{M}{2} \rfloor - N_2 = N_1$, and then $b(n)$ can be obtained by using (32). When $M > 2N_2$, $N_1 + 2N_2 = L + K - 1$, since M is odd. If we choose $N_2 = 0$, then $N_1 = L + K - 1$, which is correspondent to the FIR orthonormal solution in [9]. If we choose $N_1 = 0$, then $M = L + K$. When M is odd, $2N_2 = L + K - 1 = M - 1$, while if M is even, $2N_2 = L + K = M$. Thus $N_2 = \lfloor \frac{M}{2} \rfloor = \lfloor \frac{L+K}{2} \rfloor$, and it is the IIR orthonormal solution in [9]. Therefore, it is clear that the FIR and IIR orthonormal solutions proposed in [9] are only the special cases of general IIR orthonormal solutions when $N_2 = 0$ or $N_1 = 0$.

1) Example 1: We consider a class of DT$\mathcal{C}$WTs using IIR filters with numerator and denominator of different degree. As proposed in [9], we have used the maximally flat allpass filter with $L = 2$ and $K = 4$. To obtain the filters of minimal degree, we can choose we can choose $\{N_1, N_2\} = \{5, 0\}, \{3, 1\}, \{1, 2\}, \{0, 3\}$, where the degree of numerators are $M = 11, 9, 7, 6$, respectively. Note that the filter with $\{N_1, N_2\} = \{5, 0\}$ is FIR filter. We have designed these four filters, and the resulting magnitude responses of $H_i(z)$ are shown in Fig. 2. It is seen that IIR filters have more sharp magnitude responses than FIR filter. To get stable filters, the numerator and denominator are obtained by using the minimum-phase spectral factor [9]. Their group delay responses are given in Fig. 3. It is seen that the group delay becomes more flat as a decreasing N_2, and the half-sample delay condition is approximately achieved. Moreover, the magnitude responses of $E(\omega)$ are shown in Fig. 4. The maximum error of IIR filters are smaller than the conventional FIR filter. Furthermore, the spectrum $\Psi_i(\omega)$ and the spectrum $\Psi_c(\omega)$ are shown in Figs. 5 and 6 respectively. In Fig. 6, the complex wavelet constructed by FIR filter has a bigger spectrum in the negative frequency domain than that by IIR filters. In addition, the scaling and wavelet functions $\phi_i(t), \psi_i(t)$ are also shown in Fig. 7. Finally, the filter coefficients of $H_i(z)$ are given in Table I while the analyticity measures of E_∞ and E_2 are summarized in Table II and both of E_∞ and E_2 decrease as an increasing N_2.

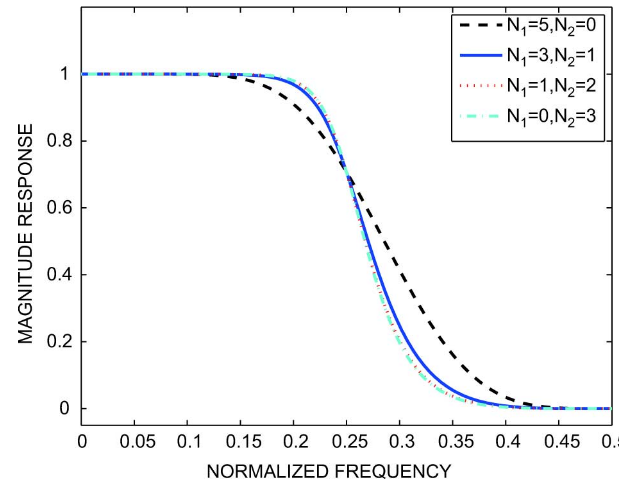

Fig. 2. Magnitude responses of scaling lowpass filters $H_i(z)$ in Example 1.

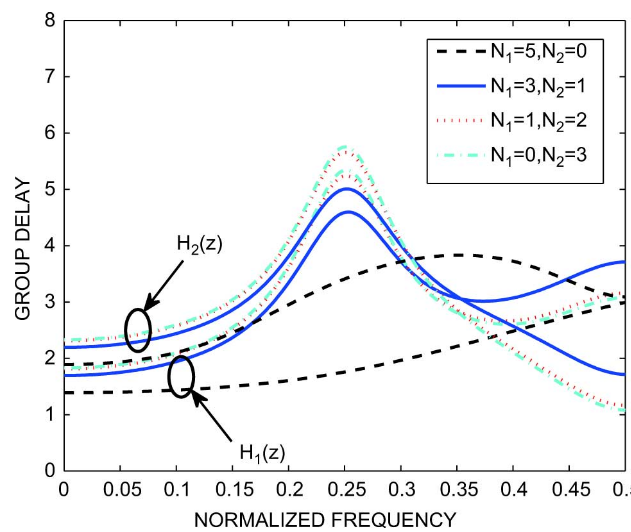

Fig. 3. Group delays of scaling lowpass filters $H_i(z)$ in Example 1.

IV. DT$\mathcal{C}$WTs With Improved Analyticity

In [9], Selesnick had used the maximally flat allpass filters for $A(z)$. Since $\omega = 0$ is chosen as the point of approximation,

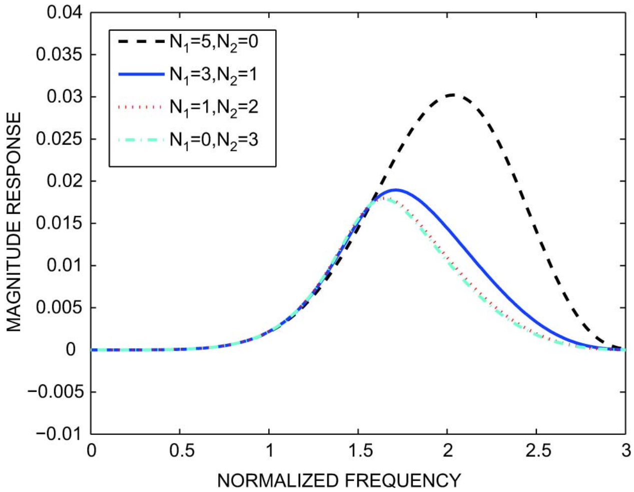

Fig. 4. Magnitude responses of $E(\omega)$ in Example 1.

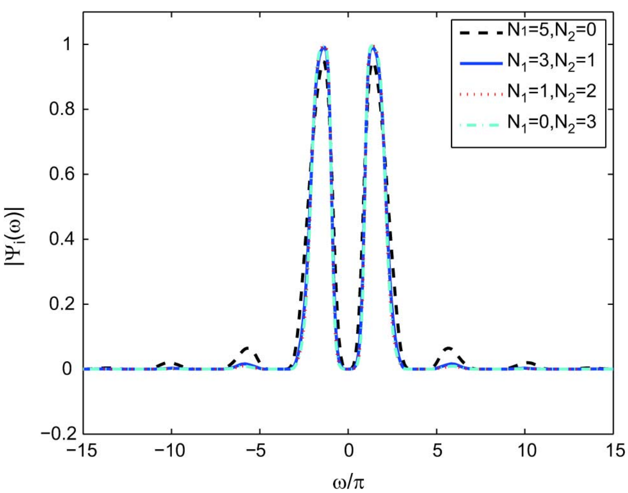

Fig. 5. Magnitude responses of $\Psi_i(\omega)$ in Example 1.

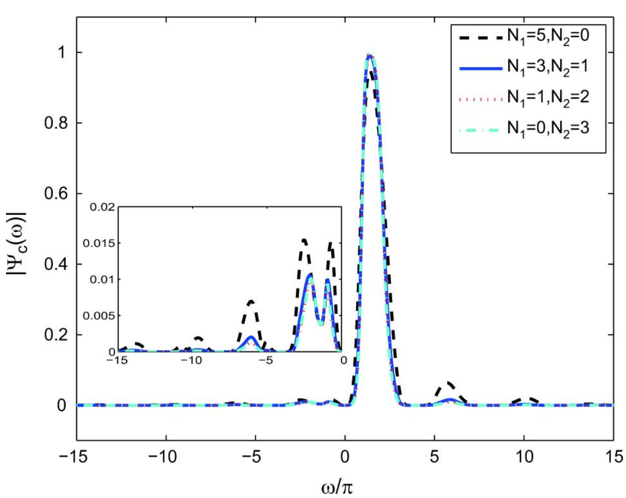

Fig. 6. Magnitude responses of $\Psi_c(\omega)$ in Example 1.

the phase error will increase as $|\omega|$ increases. Thus, $E(\omega)$ has a large error in transition band (see Fig. 4), resulting in a poor analyticity of complex wavelet. In the following, we will discuss how to improve the analyticity. From (9) and (13), we have

$$E(\omega) = H_1(e^{j\omega})[A(e^{j\omega}) - e^{-j\frac{\omega}{2}}], \tag{34}$$

thus

$$|E(\omega)| = 2\left|H_1(e^{j\omega})\right| \left|\sin\frac{\theta(\omega)+\frac{\omega}{2}}{2}\right|. \tag{35}$$

where $\theta(\omega)$ is the phase response of $A(z)$. It is clear that $|E(\omega)|$ is dependent on both the magnitude response $|H_1(e^{j\omega})|$ and the phase error of $A(z)$. Since $H_1(z)$ is a lowpass filter, we must minimize the phase error not only in passband but also in transition band to improve the analyticity of complex wavelet. There are many design methods for allpass filters to approximate a fractional delay response, for example, the maximally flat [3], equiripple approximations [4], and so on. It will be better if the minimax (Chebyshev) phase approximation of allpass filters is used, i.e., [4].

It is known that the wavelet function is defined by the infinite product formula. Thus, it is necessary that $A(z)$ has a certain degree of flatness at $\omega = 0$ to improve the analyticity. In [17], we present a design method of allpass filters with the specified degree of flatness at $\omega = 0$ and equiripple phase response in the approximation band.

The desired phase response is $\theta_d(\omega) = -\frac{1}{2}\omega$. The difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d(\omega)$ is given by

$$\theta_e(\omega) = \theta(\omega) - \theta_d(\omega) = 2\tan^{-1}\frac{N_L(\omega)}{D_L(\omega)}, \tag{36}$$

where

$$\begin{cases} N_L(\omega) = \sum\limits_{n=0}^{L} d(n)\sin\left\{\left(n-\frac{L}{2}+\frac{1}{4}\right)\omega\right\} \\ D_L(\omega) = \sum\limits_{n=0}^{L} d(n)\cos\left\{\left(n-\frac{L}{2}+\frac{1}{4}\right)\omega\right\}. \end{cases} \tag{37}$$

Therefore, the problem is to satisfy the flatness condition and minimize the phase error $\theta_e(\omega)$ in the approximation band.

Firstly, we consider the flatness condition of the phase response at $\omega = 0$. It is required that the derivatives of $\theta(\omega)$ are equal to that of $\theta_d(\omega)$ at $\omega = 0$;

$$\left.\frac{\partial^{2r+1}\theta(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = \left.\frac{\partial^{2r+1}\theta_d(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} (r = 0, 1, \ldots, J-1), \tag{38}$$

where J is a parameter that controls the degree of flatness, and $0 \le J \le L$. Equation (38) is equivalent to

$$\left.\frac{\partial^{2r+1}\theta_e(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0 (r = 0, 1, \ldots, J-1). \tag{39}$$

From (36), (39) can be reduced to

$$\left.\frac{\partial^{2r+1}N_L(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = 0\ (r = 0, 1, \ldots, J-1). \tag{40}$$

By substituting $N_L(\omega)$ in (37) into (40), we can derive a system of linear equations as follows;

$$\sum_{n=0}^{L}(n-\frac{L}{2}+\frac{1}{4})^{2r+1}d(n) = 0\ (r = 0, 1, \ldots, J-1). \tag{41}$$

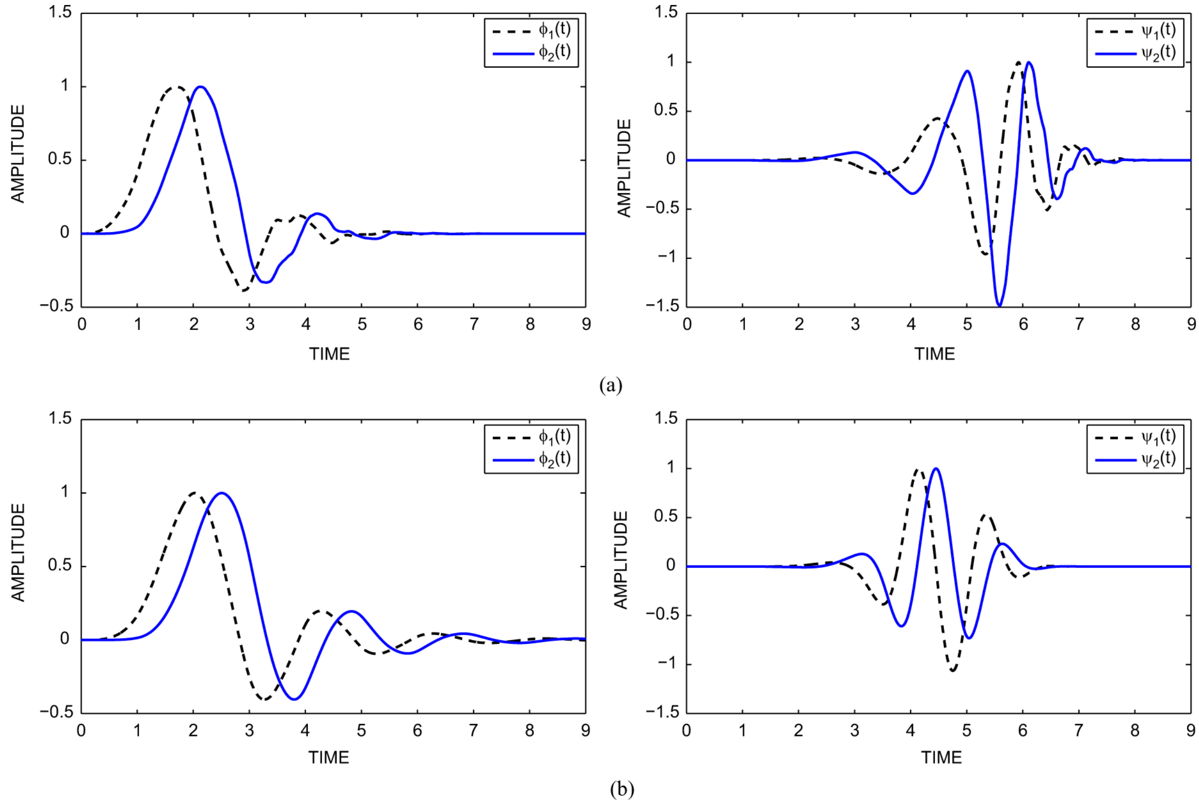

Fig. 7. Scaling and wavelet functions $\phi_i(t)$, $\psi_i(t)$ in Example 1: (a) $N_1 = 5, N_2 = 0$ (b) $N_1 = 3, N_2 = 1$.

TABLE I
FILTER COEFFICIENTS OF $H_i(z)$ WITH $N_1 = 3, N_2 = 1$ IN EXAMPLE 1

n	Numerator of $H_1(z)$	Numerator of $H_2(z)$	Denominator of $H_i(z)$
0	0.06060304	0.01212061	1
1	0.34027062	0.16501899	0
2	0.72397685	0.55347756	0.46902285
3	0.70741284	0.78974799	
4	0.27453195	0.50351744	
5	-0.01220079	0.08905209	
6	-0.02055616	-0.03278854	
7	0.00330903	-0.00488464	
8	0.00020034	0.00242895	
9	-0.00003568	-0.00017841	

TABLE II
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 1

N_1	N_2	$E_\infty(\%)$	$E_2(\%)$
5	0	1.627	1.894
3	1	1.064	1.173
1	2	1.017	1.061
0	3	1.014	1.048

Note that if $J = L$, we can solve the linear equations in (41) to obtain the maximally flat allpass filters, due to $d(0) = 1$.

Next, we consider the case of $J < L$. We want to obtain an equiripple phase response in the approximation band $[0, \omega_c]$ by using the remaining degree of freedom. Let ω_i $(0 < \omega_0 < \omega_1 < \cdots < \omega_{L-J} = \omega_c)$ be the extremal frequencies in the approximation band. We apply the Remez exchange algorithm and formulate $\theta_e(\omega)$ as

$$\tan\frac{\theta_e(\omega_i)}{2} = \frac{\sum_{n=0}^{L} d(n)\sin\left\{\left(n-\frac{L}{2}+\frac{1}{4}\right)\omega_i\right\}}{\sum_{n=0}^{L} d(n)\cos\left\{\left(n-\frac{L}{2}+\frac{1}{4}\right)\omega_i\right\}} = (-1)^i\delta, \quad (42)$$

where δ is an error. We then rewrite (41) and (42) in the matrix form as

$$\mathbf{Pd} = \delta\mathbf{Qd}, \quad (43)$$

where $\mathbf{d} = [d(0), d(1), \ldots, d(L)]^T$, and the elements of the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are given by

$$P(m,n) = \begin{cases} \left(n - \frac{L}{2} + \frac{1}{4}\right)^{(2m+1)} & (m = 0, 1, \ldots, J-1) \\ \sin\{(n - \frac{L}{2} + \frac{1}{4})\omega_{(m-J)}\} & \\ & (m = J, J+1, \ldots, L). \end{cases} \quad (44)$$

$$Q(m,n) = \begin{cases} 0 & (m = 0, 1, \ldots, J-1) \\ (-1)^{m-J}\cos\left\{\left(n - \frac{L}{2} + \frac{1}{4}\right)\omega_{(m-J)}\right\} & \\ & (m = J, J+1, \ldots, L). \end{cases} \quad (45)$$

It should be noted that (43) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\mathbf{d}$ is the corresponding eigenvector. To minimize δ, we should choose the absolute minimum eigenvalue by solving the eigenvalue problem, thus the corresponding eigenvector give a set of filter coefficients $d(n)$. To be an equiripple phase response in the

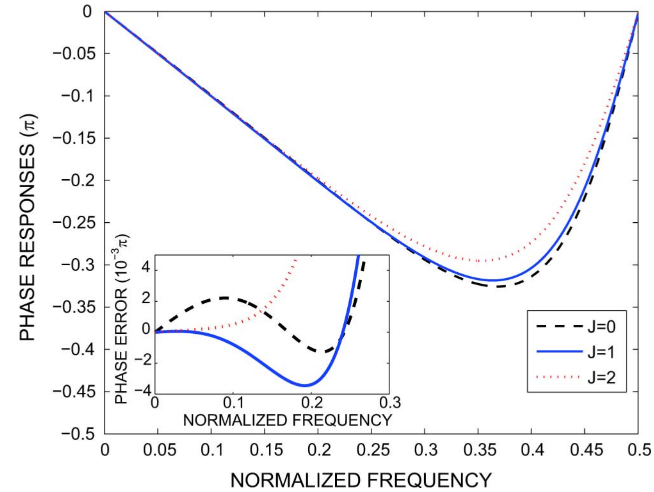

Fig. 8. Phase responses of allpass filters $A(z)$ and phase errors in the inset in Example 2.

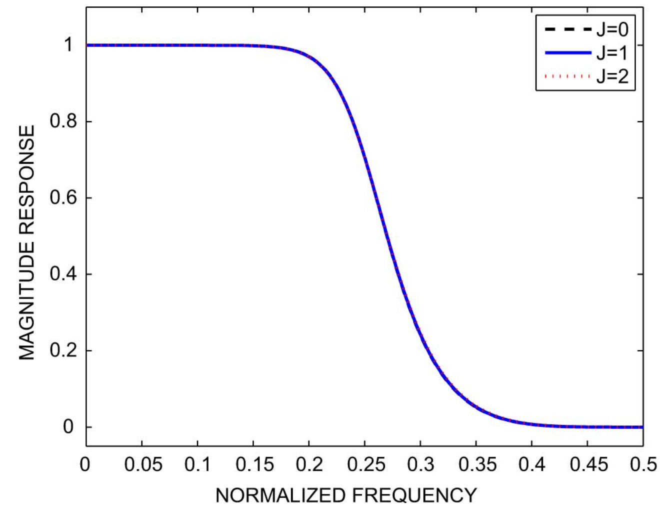

Fig. 9. Magnitude responses of scaling lowpass filters $H_i(z)$ in Example 2.

TABLE III
FILTER COEFFICIENTS OF $H_i(z)$ WITH $N_1 = 3, N_2 = 1, J = 1$ IN EXAMPLE 2

n	Numerator of $H_1(z)$	Numerator of $H_2(z)$	Denominator of $H_i(z)$
0	0.06430172	0.01469667	1
1	0.35061982	0.17226396	0
2	0.73000792	0.55903614	0.47360517
3	0.70140047	0.78908873	
4	0.26786930	0.50060387	
5	-0.01322045	0.08578928	
6	-0.02047718	-0.03510981	
7	0.00325037	-0.00490950	
8	0.00029445	0.00276934	
9	-0.00005400	-0.00023627	

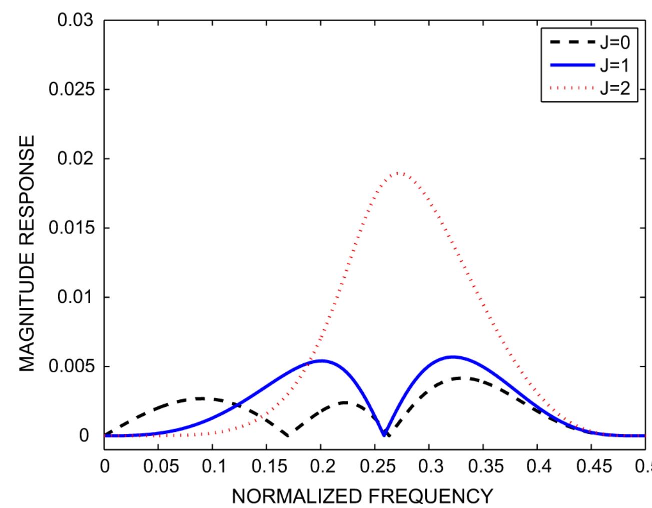

Fig. 10. Magnitude responses of $E(\omega)$ in Example 2.

approximation band, we make use of an iteration procedure to get the optimal filter coefficients $d(n)$[4], [17].

1) Example 2: We consider a class of DT$\mathcal{C}$WTs with improved analyticity. Firstly, we have designed allpass filters with $L = 2$, $J = \{0, 1, 2\}$, $\omega_c = 0.55\pi$. Note that $J = 2$ means the maximally flat allpass filter, while $J = 0$ is the equiripple allpass filter without the flatness condition proposed in [4]. The resulting phase responses of $A(z)$ are then shown in Fig. 8, where the maximally flat allpass filter has a larger phase error compared with other allpass filters. Next, we have used the method proposed in Section III-C to construct the scaling lowpass filters $H_i(z)$ with $K = 4$, $N_1 = 3$, $N_2 = 1$, where the filter coefficients of $H_i(z)$ with $A(z)$ of $J = 1$ are given in Table III. The magnitude responses of $H_i(z)$ are shown in Fig. 9 and are almost the same. The magnitude responses of $E(\omega)$ are also shown in Fig. 10, and the maximum error of $E(\omega)$ decreases as an decreasing J. However, the analyticity measures of E_∞ and E_2 become minimum when $J = 1$, not $J = 0$, as shown in Table IV. It is seen in (35) that $|E(\omega)|$ is mainly dependent on the phase error of $A(z)$ in the passband and the magnitude response of $H_1(z)$ in the stopband. Thus decreasing J reduces the maximum error of $|E(\omega)|$. However, the wavelet function is defined by the infinite product formula; $\Psi_i(\omega) = \Phi_i(0)\sqrt{2}G_i(e^{j\frac{\omega}{2}})\prod_{n=2}^{\infty}\sqrt{2}H_i(e^{j\frac{\omega}{2^n}})$, and the phase of $\psi_i(t)$ is a sum of the phase responses of $G_i(e^{j\frac{\omega}{2}})$ and $H_i(e^{j\frac{\omega}{2^n}})$ $(n = 2, 3, \ldots, \infty)$. If there is a larger phase error nearby $\omega = 0$ between $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$, it will be added to result in a poor analyticity. Therefore, a certain degree of flatness is required to reduce the phase error nearby $\omega = 0$. Finally, the spectrum $\Psi_i(\omega)$ and $\Psi_c(\omega)$ are also shown in Figs. 11 and 12, where the proposed allpass filter has improved the analyticity, compared with the maximally flat allpass filter used in [9].

TABLE IV
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 2

J	$E_\infty(\%)$	$E_2(\%)$
0	0.499	0.514
1	0.395	0.417
2	1.064	1.173

V. DT$\mathcal{C}$WTS WITH IMPROVED FREQUENCY SELECTIVITY

It is well-known that frequency selectivity is also a useful property for many applications of signal and image processing.

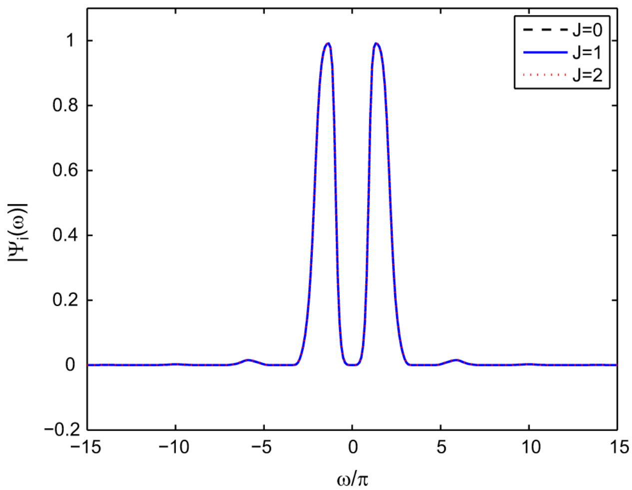

Fig. 11. Magnitude responses of $\Psi_i(\omega)$ in Example 2.

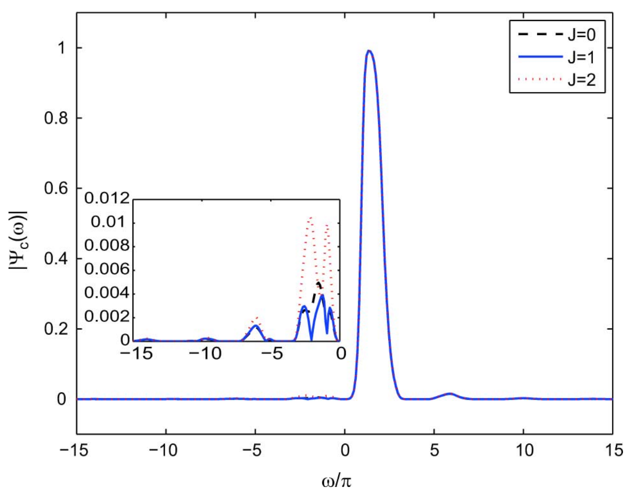

Fig. 12. Magnitude responses of $\Psi_c(\omega)$ in Example 2.

The maximally flat filters generally have a poor frequency selectivity. In the above-mentioned DT$\mathcal{C}$WTs, the scaling lowpass filters have as many zeros at $z = -1$ as possible to obtain the maximum number of vanishing moments. In [15] and [16], we have applied the Remez exchange algorithm to improve the frequency selectivity of scaling lowpass filters. In this section, we describe a design algorithm based on the eigenvalue problem.

We first specify the number of zeros at $z = -1$ for $H_i(z)$ from the viewpoint of regularity. We assume $K < K_{max}$ where $K_{max} = \lfloor \frac{M+1}{2} \rfloor + N_2 - L$. Then the remaining degree of freedom is $K_{max} - K$. Since zeros on the unit circle are complex-conjugate pair except $z = \pm 1$, $K_{max} - K$ should be even, i.e., $K_{max} - K = 2I$.

Next, we apply the Remez exchange algorithm in the stopband $[\omega_s, \pi]$ to get an equiripple magnitude response. We set ω_i ($\omega_s = \omega_0 < \omega_1 < \cdots < \omega_{2I} < \pi$) to be a set of extremal frequencies and formulate $P(e^{j\omega})$ as

$$P(e^{j\omega_i}) = \frac{R(e^{j\omega_i})S(e^{j\omega_i})}{B(e^{j2\omega_i})} = \frac{1+(-1)^i}{2}\delta, \tag{46}$$

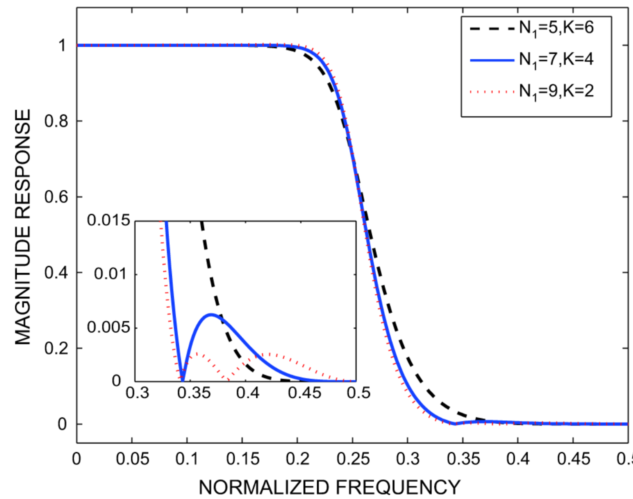

Fig. 13. Magnitude responses of scaling lowpass filters $H_i(z)$ in Example 3.

where $\delta(> 0)$ is an error. Note that we force $P(e^{j\omega_i}) \geq 0$ to permit spectral factorization of $R(e^{j\omega_i})$. From (46), we have

$$S(e^{j\omega_i})R(e^{j\omega_i}) = \frac{1+(-1)^i}{2}\delta B(e^{j2\omega_i}), \tag{47}$$

where

$$\begin{cases} R(e^{j\omega_i}) = r(0) + 2\sum_{n=1}^{N_1} r(n)\cos(n\omega) \\ B(e^{j\omega_i}) = b(0) + 2\sum_{n=1}^{N_2} b(n)\cos(n\omega). \end{cases} \tag{48}$$

Thus, we rewrite (47) in the matrix form as

$$\mathbf{P_1 r} = \delta \mathbf{Q_1 b}, \tag{49}$$

where the elements of the matrices $\mathbf{P_1}$ and $\mathbf{Q_1}$ are given by

$$P_1(m,n) = \begin{cases} S(e^{j\omega_m}) & (n=0) \\ 2S(e^{j\omega_m})\cos(n\omega_m) & (n = 1, 2, \ldots, N_1) \end{cases}$$

$$Q_1(m,n) = \begin{cases} \frac{(1+(-1)^m)}{2} & (n=0) \\ (1+(-1)^m)\cos(2n\omega_m) & (n = 1, 2\ldots, N_2). \end{cases}$$

It should be noted that the orthonormality condition has been given in (32) and (33). Hence, we use (32) to obtain $\mathbf{P_1 r} = \delta \mathbf{Q_1 b} = \delta \mathbf{Q_1 S_1 r}$. Then we have

$$\begin{bmatrix} \mathbf{S_2} \\ \mathbf{P_1} \end{bmatrix} \mathbf{r} = \delta \begin{bmatrix} \mathbf{0} \\ \mathbf{Q_1 S_1} \end{bmatrix} \mathbf{r}, \tag{50}$$

which is correspondent to a generalized eigenvalue problem. We choose the minimum positive eigenvalue δ and the corresponding eigenvector gives a set of filter coefficients $r(n)$. By making use of an iteration procedure, we can obtain the optimal filter coefficients $r(n)$. We then compute $b(n)$ by (32).

1) Example 3: We consider a class of DT$\mathcal{C}$WTs with improved analyticity and frequency selectivity. Firstly, we have used the allpass filter with $L = 2$, $J = 1$, $\omega_c = 0.51\pi$, and then designed the scaling lowpass filters $H_i(z)$ with $M = 13$, $N_2 = 1$, $\omega_s = 0.67\pi$. We set $K = 4$ and $N_1 = 7$. The resulting magnitude response of $H_i(z)$ is shown in Fig. 13. For

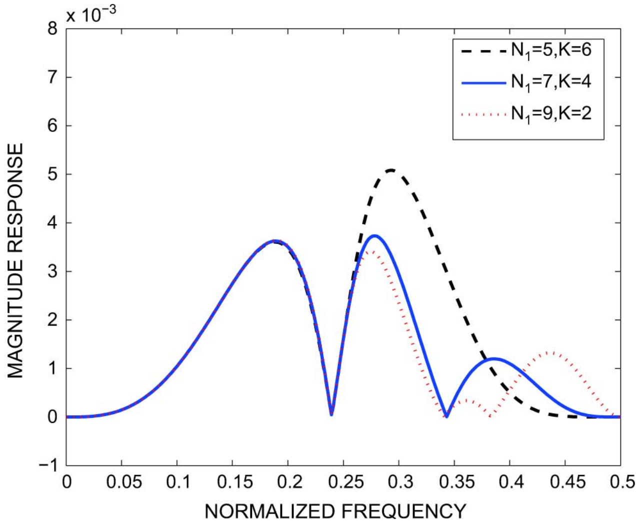

Fig. 14. Magnitude responses of $E(\omega)$ in Example 3.

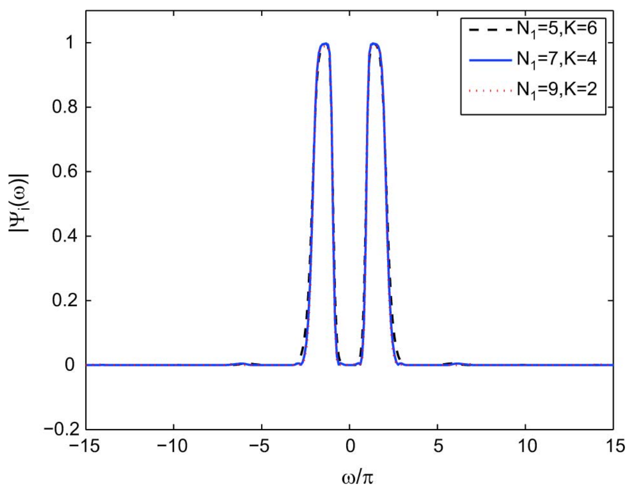

Fig. 15. Magnitude responses of $\Psi_i(\omega)$ in Example 3.

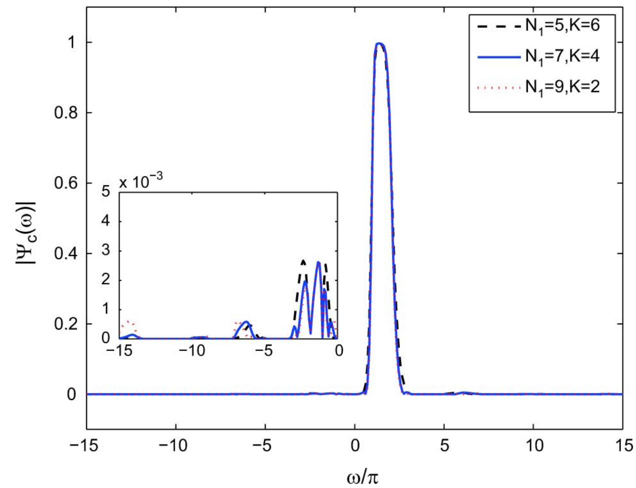

Fig. 16. Magnitude responses of $\Psi_c(\omega)$ in Example 3.

TABLE V
FILTER COEFFICIENTS OF $H_i(z)$ WITH $N_1 = 7$, $N_2 = 1$, $K = 4$ IN EXAMPLE 3

n	Numerator of $H_1(z)$	Numerator of $H_2(z)$	Denominator of $H_i(z)$
0	0.04097186	0.00918649	1
1	0.24569827	0.11481153	0
2	0.6058383	0.42067653	0.61517537
3	0.7893763	0.74485096	
4	0.54962226	0.71840965	
5	0.13301111	0.33505370	
6	-0.06721645	-0.00614984	
7	-0.02964467	-0.06349930	
8	0.01518836	0.00117825	
9	0.00290343	0.01156778	
10	-0.00273125	-0.00196788	
11	0.00037645	-0.00104015	
12	0.00004097	0.00038085	
13	-0.00000683	-0.00003047	

TABLE VI
ANALYTICITY MEASURES E_∞ AND E_2 IN EXAMPLE 3

K	N_1	$E_\infty(\%)$	$E_2(\%)$
6	5	0.268	0.299
4	7	0.262	0.236
2	9	0.258	0.232

comparison, the scaling lowpass filter with the maximally flat magnitude response ($K = 6,\ N_1 = 5$), and the filter with two equiripples in the stopband ($K = 2,\ N_1 = 9$) are also designed and their magnitude responses are shown in Fig. 13. It is clear that the magnitude responses of $H_i(z)$ with improved frequency selectivity are more sharp than the maximally flat filter. In addition, the magnitude responses of $E(\omega)$ are shown in Fig. 14, where the error decreases at the expense of decreasing vanishing moments. The wavelet spectrum $\Psi_i(\omega)$ are shown in Fig. 15, which are almost same. Furthermore, the spectrum $\Psi_c(\omega)$ are shown in Fig. 16, where the spectrum is close to zero in the negative frequency domain. Besides, the scaling functions $\phi_i(t)$ and wavelet functions $\psi_i(t)$ are shown in Fig. 17. Finally, the filter coefficients of $H_i(z)$ with one equiripple in the stopband are given in Table V. Table VI summarizes the analyticity measures of E_∞ and E_2. It is seen that the analyticity can be also improved slightly by improving the frequency selectivity of $H_i(z)$.

VI. PERFORMANCE INVESTIGATION

In this section, we present several design examples to investigate the performance on the proposed DT$\mathcal{C}$WTs with improved analyticity and frequency selectivity. First of all, we have designed the allpass filters with $L = 2$, $J = 1$ and the cutoff frequency is chosen as $\omega_c = \{0.35\pi, 0.52\pi, 0.80\pi\}$. Then we have constructed the scaling lowpass filters $H_i(z)$ with $K = 2$, $N_1 = 5$, $N_2 = 1$ and $\omega_s = 0.67\pi$. The resulting magnitude responses of $E(\omega)$ are shown in Fig. 18, and it is seen that the maximum error of $E(\omega)$ is the minimum when $\omega_c = 0.52\pi$. If ω_c is chosen to be too small or too big, the maximum error of $E(\omega)$ will increase, resulting in a poor analyticity. That is to say, how to determine the cutoff frequency ω_c influences $E(\omega)$ as well as the analyticity of complex wavelets. Next, we have varied ω_c from 0.3π to 0.8π to investigate the relationship between the analyticity measures of E_∞, E_2 and the cutoff frequency ω_c. It is seen in Fig. 19 when ω_c is too small or too big, the analyticity measures of E_∞, E_2 become larger, and the optimal

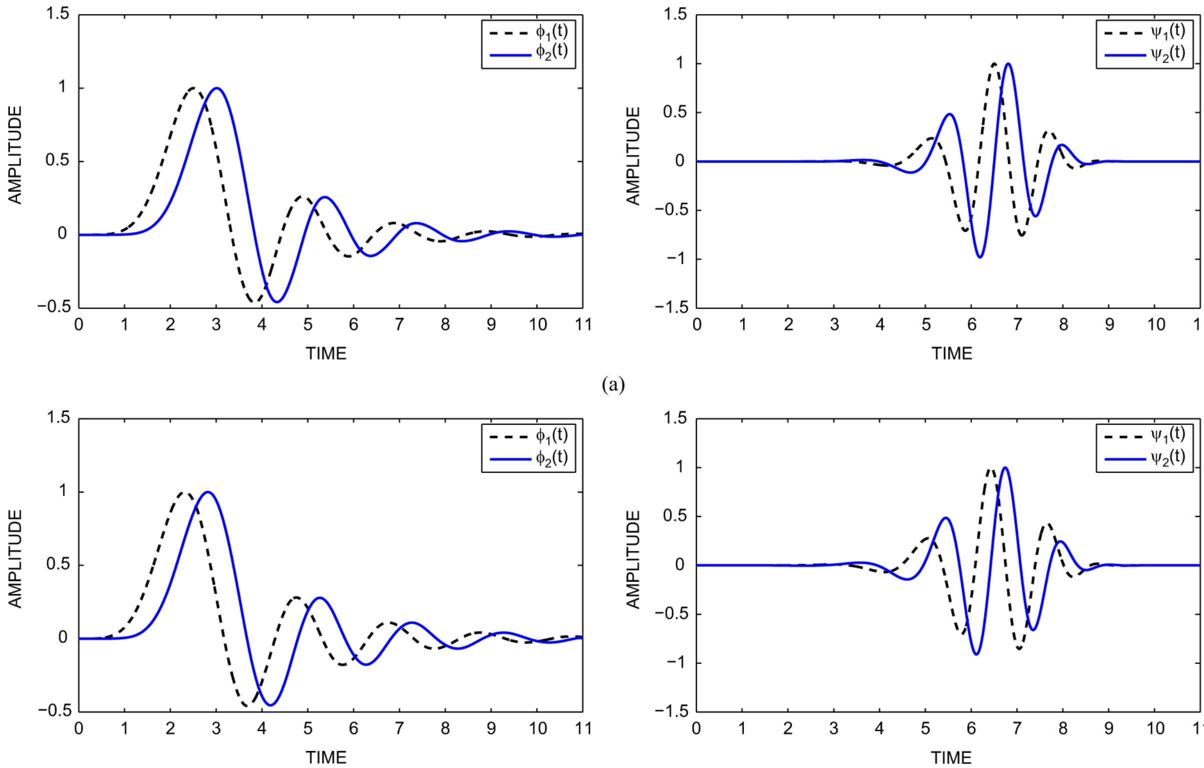

Fig. 17. Scaling and wavelet functions $\phi_i(t), \psi_i(t)$ in Example 3 (a) $N_1 = 5, K = 6$ (b) $N_1 = 7, K = 4$.

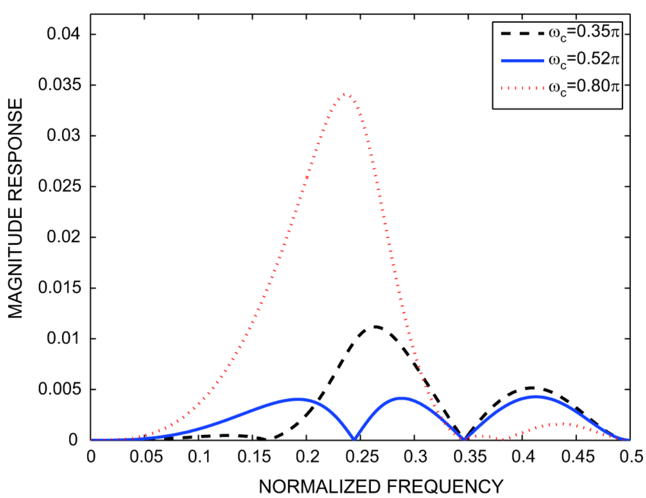

Fig. 18. Magnitude responses of $E(\omega)$.

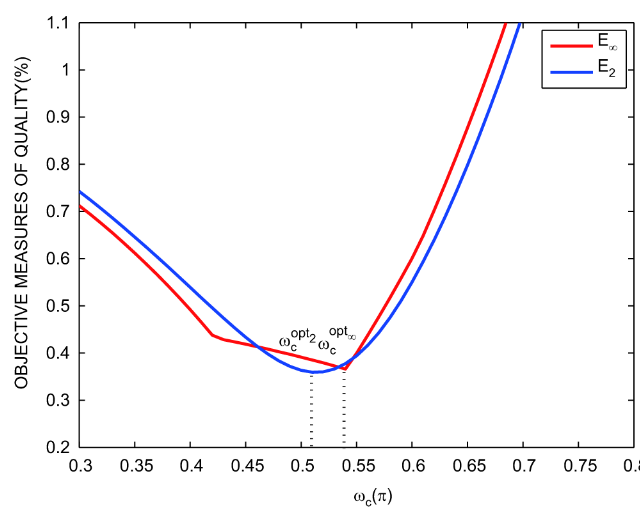

Fig. 19. Relationship between E_∞, E_2 and ω_c.

cutoff frequencies $\omega_c^{opt_\infty}$ and $\omega_c^{opt_2}$ are within $(0.52\pi, 0.55\pi)$. Furthermore, we have also varied ω_s from 0.55π to 0.95π to investigate the relationship between the optimal cutoff frequency ω_c^{opt} and ω_s. It is clear in Fig. 20 that if the stopband is too wide, i.e., ω_s is closer to 0.5π, the optimal cutoff frequency ω_c^{opt} is larger, that is, the approximation band of allpass filter become wider. It is because the stopband error of lowpass scaling filter is larger in this case, requiring the allpass filter to improve the error $E(\omega)$ also in stopband. On other hand, the optimal cutoff frequency ω_c^{opt} is almost constant when the stopband band is not too wide, since the stopband error of lowpass scaling filter is small and has little effect on the analyticity.

VII. Conclusion

In this paper, we have proposed a new class of DT$\mathcal{C}$WTs with improved analyticity and frequency selectivity by using general IIR filters with numerator and denominator of different degree. The proposed DT$\mathcal{C}$WTs include the conventional DT$\mathcal{C}$WTs proposed by Selesnick as special cases. First of all, we have given a design method of allpass filters with the specified degree of flatness and equiripple phase responses in the approximation band to improve the analyticity of complex wavelets. Next, we have specified the number of vanishing moments and applied the Remez exchange algorithm to minimize the stopband error

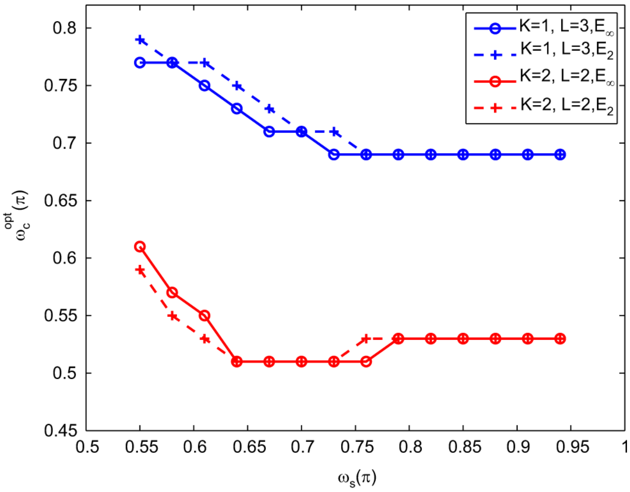

Fig. 20. Relationship between ω_c^{opt} and ω_s.

in order to improve the frequency selectivity of scaling lowpass filters. Finally, we have done the performance investigation on the proposed $\mathrm{DT}\mathcal{C}\mathrm{WTs}$, where a properly chosen approximation band can improve the analyticity of complex wavelets.

REFERENCES

[1] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: SIAM, 1992.

[2] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[3] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay: Tools for fractional delay filter design," *IEEE Signal Process. Mag.*, vol. 13, no. 1, pp. 30–60, Jan. 1996.

[4] X. Zhang and H. Iwakura, "Design of IIR digital allpass filters based on eigenvalue problem," *IEEE Trans. Signal Process.*, vol. 47, no. 2, pp. 554–559, Feb. 1999.

[5] N. G. Kingsbury, "The dual-tree complex wavelet transform: A new technique for shift invariance and directional filters," presented at the 8th IEEE Digit. Signal Process. (DSP) Workshop, Bryce Canyon National Park, UT, Aug. 1998.

[6] N. G. Kingsbury, "A dual-tree complex wavelet transform with improved orthogonality and symmetry properties," in *Proc. Int. Conf. Image Process. (ICIP)*, Vancouver, Canada, Sep. 2000, vol. 2, pp. 375–378.

[7] N. G. Kingsbury, "Complex wavelets for shift invariant analysis and filtering of signals," *Appl. Comput. Harmon. Anal.*, vol. 10, no. 3, pp. 234–253, May 2001.

[8] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," *IEEE Signal Process. Lett.*, vol. 8, no. 6, pp. 170–173, Jun. 2001.

[9] I. W. Selesnick, "The design of approximate Hilbert transform pairs of wavelets bases," *IEEE Trans. Signal Process.*, vol. 50, no. 5, pp. 1144–1522, May 2002.

[10] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The dual-tree complex wavelet transform," *IEEE Signal Process. Mag.*, vol. 22, no. 6, pp. 123–151, Nov. 2005.

[11] H. Ozkaramanli and R. Yu, "On the phase condition and its solution for Hilbert transform pairs of wavelet bases," *IEEE Trans. Signal Process.*, vol. 51, no. 12, pp. 3293–3294, Dec. 2003.

[12] R. Yu and H. Ozkaramanli, "Hilbert transform pairs of orthogonal wavelet bases: Necessary and sufficient conditions," *IEEE Trans. Signal Process.*, vol. 53, no. 12, pp. 4723–4725, Dec. 2005.

[13] D. B. H. Tay, N. G. Kingsbury, and M. Palaniswami, "Orthonormal Hilbert pair of wavelets with (almost) maximum vanishing moments," *IEEE Signal Process. Lett.*, vol. 13, no. 9, pp. 533–536, Sep. 2006.

[14] D. B. H. Tay, "A new approach to the common-factor design technique for Hilbert-pair of wavelets," *IEEE Signal Process. Lett.*, vol. 17, no. 11, pp. 969–972, Nov. 2010.

[15] X. Zhang, "Design of Hilbert transform pairs of orthonormal wavelet bases using Remez exchange algorithm," in *Proc. Int. Conf. Image Process. (ICIP)*, Cairo, Egypt, Nov. 2009, pp. 3813–3816.

[16] D. W. Wang and X. Zhang, "Design of Hilbert transform pairs of orthonormal wavelet bases using IIR filters," in *Proc. Int. Symp. Commun. Inf. Technol. (ISCIT)*, Tokyo, Japan, Oct. 2010, pp. 554–559.

[17] X. Zhang, "Design of Hilbert transform pairs of orthonormal wavelet bases with improved analyticity," in *Proc. Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Prague, Czech Republic, May 2011, pp. 4384–4387.

[18] D. W. Wang and X. Zhang, "Performance investigation on $\mathrm{DT}\mathcal{C}\mathrm{WT}$ based on the common factor technique," presented at the Int. Conf. Acoust., Speech, Signal Process. (ICASSP), Kyoto, Japan, Mar. 2012.

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