Design of IIR Digital Filters Based on Eigenvalue Problem

Xi Zhang, *Member, IEEE*, and Hiroshi Iwakura, *Member, IEEE*

***Abstract*— This paper presents a new method for designing IIR digital filters with optimum magnitude response in the Chebyshev sense and different order numerator and denominator. The proposed procedure is based on the formulation of a generalized eigenvalue problem by using Remez exchange algorithm. Since there exist more than one eigenvalue in the general eigenvalue problem, we introduce a very simple selection rule for the eigenvalue to be sought for where the rational interpolation is performed if and only if the positive minimum eigenvalue is chosen. Therefore, the solution of the rational interpolation problem can be obtained by computing only one eigenvector corresponding to the positive minimum eigenvalue, and the optimal filter coefficients are easily obtained through a few iterations. The design algorithm proposed in this paper not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the positive minimum eigenvalue. Some properties of the filters such as lowpass filters, bandpass filters, and so on are discussed, and several design examples are presented to demonstrate the effectiveness of this method.**

I. Introduction

A LARGE number of procedures are available for designing infinite impulse response (IIR) digital filters [1]–[15]. Some of them, like the bilinear transformation and impulse invariant design [1]–[5], transform a given analog filter into an equivalent digital filter. However, these techniques are limited in that they are generally applied only to the case of transforming standard analog filters. If it is desired to design a digital filter with a nonstandard frequency response that cannot be obtained using some of the above techniques, then it is necessary to use some other procedures.

Design methods using linear programming and the differential correction algorithm are described in [6]–[8]. Both methods found the optimum rational approximation to the squared magnitude response instead of the magnitude alone. This is because in the form of squared magnitude, the design problem can be linearized, making it possible to apply standard linear optimization techniques. Deczky [9] described a very general design method that is solved by using the Fletcher–Powell algorithm. This method often requires a large amount of computer time, even for a moderate size problem. Variations of the Remez exchange algorithm [10]–[15] have also been applied to the rational approximation problem. The Remez exchange algorithm, while potentially the fastest of the above procedures, suffers from some serious problems that are not encountered in finite impulse response (FIR) applications. Most of the difficulty is in the central part of the Remez exchange algorithm, where a rational interpolation problem must be solved to make the error oscillate with an equal amplitude on the trial set of extremal frequencies. In [10] and [13], the rational interpolation problem was solved by using the Newton method. In [12], restricting all zeros of the filters on the unit circle, the algorithm used two approximation intervals and worked separately with the numerator and denominator polynomials of the squared magnitude function. In [15], Jackson presented an improved algorithm of [12] that allows some real zeros inside the unit circle.

In this paper, we consider design of IIR digital filters with different order numerator and denominator, which are more effective than elliptic filters in narrowband and wideband applications [13]. Our purpose is to develop a new design method based on the eigenvalue problem for IIR digital filters with optimum magnitude response in the Chebyshev sense. By applying the Remez exchange algorithm to the squared magnitude function, we formulate the design problem in the form of a generalized eigenvalue problem, which was already proposed by Werner [17] in 1963. There exist more than one eigenvalue in the general eigenvalue problem; therefore, we must seek out one eigenvalue that corresponds to the solution of the rational interpolation problem. However, Werner did not give a selection rule for the eigenvalue to be sought out. In this paper, we introduce a new and very simple selection rule wherein the rational interpolation is performed if and only if the positive minimum eigenvalue is chosen. Therefore, we can obtain the solution of the rational interpolation problem by finding only one eigenvector corresponding to the positive minimum eigenvalue. In order to obtain an equiripple magnitude response, we make use of an iteration procedure to get the optimal filter coefficients. The new algorithm proposed in this paper not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the positive minimum eigenvalue. In general, the design algorithm converges rapidly with a few iterations and computes efficiently without any initial guess of the solution. Some properties of the filters such as lowpass filters, bandpass filters, and so on, are described, and several examples are designed to demonstrate the effectiveness of this method.

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The authors are with the Department of Communications and Systems, University of Electro-Communications, Tokyo 182, Japan (e-mail: zhangxi@qr.cas.uec.ac.jp).

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II. Statement of the Design Problem

Let $H(z)$ be the transfer function of an IIR digital filter with numerator degree N and denominator degree M

$$H(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{\sum_{i=0}^{M} b_i z^{-i}} \tag{1}$$

where filter coefficients a_i, b_i are real, and $b_0 = 1$. The squared magnitude function of $H(z)$ is obtained by evaluating

$$H(z)H(z^{-1}) = \frac{\sum_{i=-N}^{N} c_i z^{-i}}{\sum_{i=-M}^{M} d_i z^{-i}} \tag{2}$$

where $c_i = c_{-i}(i = 1, 2, \cdots, N)$, and $d_i = d_{-i}(i = 1, 2, \cdots, M)$ along the unit circle, giving

$$|H(e^{j\omega})|^2 = \frac{N(\omega)}{D(\omega)} = \frac{c_0 + 2\sum_{i=1}^{N} c_i \cos i\omega}{d_0 + 2\sum_{i=1}^{M} d_i \cos i\omega}. \tag{3}$$

Without any loss in generality, d_0 can be set to 1. Equation (3) shows that the squared magnitude function is a ratio of trigonometric polynomials, whose numerator and denominator polynomial are linear in the unknown filter coefficients c_i and d_i. Hence, we take advantage of the squared magnitude function of (3) to formulate the filter approximation problem. Now, we state the design problem of IIR digital filters. When numerator degree N and denominator degree M are given, and the desired magnitude response $|H_d(e^{j\omega})|$ is specified in the interest bands $R \in [0, \pi]$ (e.g., passband and stopband), the aim is to find a set of filter coefficients to minimize the maximum error between the squared magnitude response of the filter and the desired squared magnitude response:

$$\text{Min}\left\{ \underset{\omega \in R}{\text{Max}} \left| W(\omega)\left[\left|H(e^{j\omega})\right|^2 - \left|H_d(e^{j\omega})\right|^2 \right] \right| \right\} \tag{4}$$

where $W(\omega)$ is a weighting function. The filter coefficients obtained in the above criterion are optimal in the Chebyshev sense. Once the filter coefficients c_i and d_i are known, it is necessary to find a stable transfer function $H(z)$. $H(z)$ has a minimum phase response that can be completely determined by retaining only zeros and poles that lie inside or on the unit circle (see [6] and [12] in detail).

Suppose that $0 \leq |H(e^{j\omega})| \leq 1$ is satisfied; then, there must exist a filter $G(z)$ that satisfies

$$\left|G(e^{j\omega})\right|^2 + \left|H(e^{j\omega})\right|^2 = 1. \tag{5}$$

In other words, $H(z)$ and $G(z)$ constitute a power-complementary pair. In particular, if $H(z)$ is a lowpass filter, then $G(z)$ is a highpass filter, and vice versa. Hence, the squared magnitude response of $G(z)$ can be obtained from (5):

$$\left|G(e^{j\omega})\right|^2 = \frac{(d_0 - c_0) + 2\sum_{i=1}^{L}(d_i - c_i)\cos i\omega}{d_0 + 2\sum_{i=1}^{M} d_i \cos i\omega} \tag{6}$$

where $L = \text{Max}\{N, M\}, d_i = 0(i = M+1, \cdots, N)$ when $N > M$, and $c_i = 0(i = N+1, \cdots, M)$ when $N < M$. It is clear from (3) and (6) that $H(z)$ and $G(z)$ have the same poles, whereas $H(z)$ has fewer zeros than $G(z)$ when $N < M$ and the same number of zeros when $N \geq M$. Therefore, we can obtain simultaneously a complementary filter pair $H(z)$ and $G(z)$ by designing only one filter $H(z)$.

III. Formulation Based on the Eigenvalue Problem

In this section, we describe the design of IIR digital filters based on the eigenvalue problem. In the above design problem, we want to find the filter coefficients c_i and d_i in (3) in such a way that the squared magnitude function with a positive denominator satisfies

$$\begin{aligned} -\delta_{\max} &\leq W(\omega)E(\omega) \\ &= W(\omega)\left[\left|H(e^{j\omega})\right|^2 - \left|H_d(e^{j\omega})\right|^2\right] \leq \delta_{\max} \quad (\omega \in R) \end{aligned} \tag{7}$$

where $E(\omega)$ is an error function, and $\delta_{\max}(>0)$ is the maximum error to be minimized.

To solve the magnitude Chebyshev approximation problem, we utilize the Remez exchange algorithm and formulate the condition for $|H(e^{j\omega})|^2$ of (3) in the form of a generalized eigenvalue problem. By selecting extremal frequencies $\omega_i(i = 1, 2, \cdots, N+M+2)$ in the bands R, we formulate $|H(e^{j\omega})|^2$ as

$$W(\omega_i)E(\omega_i) = (-1)^{(i+l)}\delta \tag{8}$$

where $l = 0$ or 1 to guarantee $\delta > 0$, and the denominator polynomial $D(\omega)$ must satisfy the following condition:

$$D(\omega) \neq 0 \quad (\text{for all } \omega). \tag{9}$$

Substituting (3) into (8), we get

$$N(\omega_i) - |H_d(e^{j\omega_i})|^2 D(\omega_i) = \frac{(-1)^{(i+l)}\delta}{W(\omega_i)} D(\omega_i). \tag{10}$$

Then, we rewrite (10) in the matrix form as

$$[\mathbf{P} - \delta\mathbf{Q}]\mathbf{A} = \mathbf{0} \tag{11}$$

where $\mathbf{A} = [c_0, c_1, \cdots, c_N, d_0, d_1, \cdots, d_M]^T, \mathbf{0} = [0, 0, \cdots, 0]^T$, and the elements of the matrices $\mathbf{P}, \mathbf{Q}$ are given by (12) and (13), which appear at the bottom of the next page. Once the desired magnitude response $|H_d(e^{j\omega})|$ and the weighting function $W(\omega)$ are given, it is seen from (12) and (13) that the elements of the matrices $\mathbf{P}, \mathbf{Q}$ are known. Therefore, it should be noted that (11) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\mathbf{A}$ is a corresponding eigenvector. It is well known that there is a nontrivial solution $\mathbf{A}(\neq\mathbf{0})$ in (11) if and only if the determinant satisfies

$$|\mathbf{P} - \delta\mathbf{Q}| = 0. \tag{14}$$

Since the order of the matrices $\mathbf{P}, \mathbf{Q}$ is $(N+M+2) \times (N+M+2)$, (14) has more than one solution to δ in general. Therefore, we can obtain at least two solutions by solving the eigenvalue problem of (11). In order to guarantee the stability of the filters and minimize the maximum magnitude error, the filter coefficients must satisfy the condition of (9). However, it is not guaranteed that the solutions obtained from

(11) have satisfied (9), that is, the solutions may not satisfy (9). The solutions that do not satisfy (9) are not required. Therefore, we must seek out the solution that satisfies (9) in the solutions of (11). Here, will we ask whether (11) has a solution that satisfies (9). If it exists, which eigenvalue corresponds to the solution? We see from (8) that the sign change of $E(\omega)$ is caused by the sign change of the numerator or denominator polynomial. When the numerator polynomial changes its sign, $E(\omega)$ crosses 0 to change its sign. When the denominator polynomial changes its sign, $E(\omega)$ crosses ∞. Therefore, there exist more than one solution, depending on the sign change of $E(\omega)$ through 0 or ∞. To satisfy (9), $E(\omega)$ must change its sign through 0. When the optimum Chebyshev approximation to the desired response exists, there are $(N+M+2)$ extremal frequencies of $E(\omega)$ [10], [18], [19]. Hence, (11) has at least one solution that satisfies (9) if the extremal frequencies are appropriately selected. By the uniqueness of the optimal solution, the solution is unique. How to select the extremal frequencies in the cases of lowpass filters, bandpass filters, and so on will be discussed in the following section. Now, we answer the second question. In (8), we can choose $l=0$ or 1 to guarantee the solution that satisfies (9) having a positive error δ. Therefore, we seek only the positive eigenvalues.

Theorem I: The positive minimum eigenvalue corresponds to the solution that satisfies (9) when the optimum Chebyshev approximation exists.

Proof: Let $\tilde{H}_1(\omega) = |H_1(e^{j\omega})|^2$ be the solution with $\delta_1(>0)$ that satisfies (9), let $\tilde{H}_2(\omega) = |H_2(e^{j\omega})|^2$ be another solution with $\delta_2(>0)$ that does not satisfy (9), and let $\hat{H}(\omega) = \tilde{H}_1(\omega) - \tilde{H}_2(\omega) = E_1(\omega) - E_2(\omega)$.

a) Assume that $\delta_1 = \delta_2$; therefore, we have $\hat{H}(\omega_i) = 0$. From (8), then $\hat{H}(\omega)$ has $(N+M+2)$ zeros in $[0,\pi]$. However, $\hat{H}(\omega)$ has at most $(N+M)$ zeros in $[0,\pi]$. Therefore, we can conclude that $\delta_1 \neq \delta_2$.

b) Assume that $\delta_1 > \delta_2$. It is seen in Fig. 1 that $\hat{H}(\omega)$ has one zero in the interval $[\omega_i, \omega_{i+1}]$ when $E_2(\omega)$ crosses 0 to change its sign and two zeros when $E_2(\omega)$ crosses ∞. There are $(N+M+1)$ interpolated intervals in $[0,\pi]$ (including the transition band). We suppose that there are I intervals where $E_2(\omega)$ changes its sign through ∞; hence, $\hat{H}(\omega)$ has $(N+M+I+1)$ zeros in $[0,\pi]$. However, $\hat{H}(\omega)$ has at most $(N+M)$ zeros in $[0,\pi]$. Hence, we can conclude that $\delta_1 < \delta_2$. The theorem is proved.

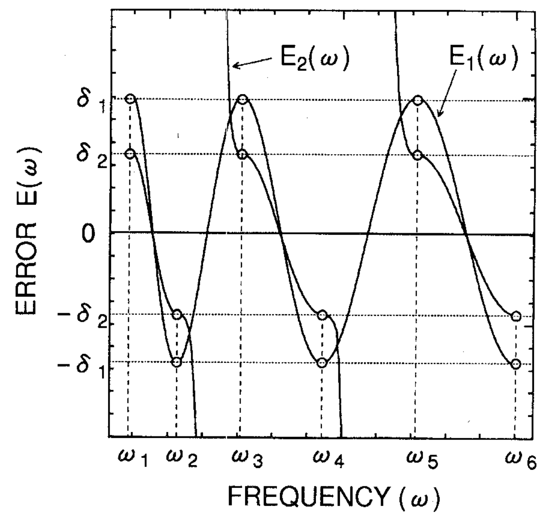


Fig. 1. Interpolation of $E(\omega)$.

We have proved that the positive minimum eigenvalue corresponds to the solution that satisfies (9). Therefore, we can obtain the solution that satisfies (9) by finding the eigenvector corresponding to the positive minimum eigenvalue. When the matrix **P** is a singular matrix, we can get $\delta = 0$ from (14). Hence, a solution can be obtained by solving the linear equations $\mathbf{PA} = \mathbf{0}$. If the solution satisfies (9), then we have obtained the desired response. However, it is generally impossible to obtain the desired response in the practical design problem. Therefore, the matrix **P** is a nonsingular matrix in the general case. Since we are interested in only one eigenvector corresponding to the positive minimum eigenvalue, we have found that the positive minimum eigenvalue is the absolute minimum one in a large number of examples; therefore, this computation can be done efficiently by using the iterative power method [20] (without invoking general methods such as the QR technique). In some other examples (e.g., lowpass filters with $N > M$ and M is odd), the absolute minimum eigenvalue is negative, and we have to use other methods to obtain the positive minimum eigenvalue. In order to obtain an equiripple magnitude response, we make use of an iteration procedure to get the optimal filter coefficients. Since we have obtained the solution that satisfies (9), we assume that the denominator polynomial is positive without any loss in generality; therefore, we can consider it to be a weighting

$$P_{ij} = \begin{cases} 1 & j = 1 \\ 2\cos(j-1)\omega_i & j = 2,3,\cdots,N+1 \\ -|H_d(e^{j\omega_i})|^2 & j = N+2 \\ -2|H_d(e^{j\omega_i})|^2\cos(j-N-2)\omega_i & j = N+3,\cdots,N+M+2 \end{cases} \tag{12}$$

$$Q_{ij} = \begin{cases} 0 & j = 1,2,\cdots,N+1 \\ \frac{(-1)^{(i+l)}}{W(\omega_i)} & j = N+2 \\ 2\frac{(-1)^{(i+l)}}{W(\omega_i)}\cos(j-N-2)\omega_i & j = N+3,\cdots,N+M+2 \end{cases} \tag{13}$$

function in the FIR applications. Therefore, the algorithm converges in general with a few iterations that are the same as the design of the FIR filters. The design algorithm is shown as follows:

Procedure {Design Algorithm of IIR Digital Filters}

Begin

1) Read numerator and denominator degree N, M, the desired magnitude response $|H_d(e^{j\omega})|$, and weighting function $W(\omega)$.
2) Select initial extremal frequencies $\Omega_i(i = 1, 2, \cdots, N + M + 2)$ equally spaced in the interest bands R.

Repeat

3) Set $\omega_i = \Omega_i(i = 1, 2, \cdots, N + M + 2)$.
4. Compute $\mathbf{P}, \mathbf{Q}$ by using (12) and (13), then find the positive minimum eigenvalue to obtain the filter coefficients c_i and d_i that satisfies (9).
5) Search the peak frequencies $\hat{\omega}_i(i = 1, 2, \cdots, J)$ of the error function $E(\omega)$ within R.
6) Reject the $(J - N - M - 2)$ superfluous peak frequencies and store the remaining frequencies into the corresponding Ω_i.

Until Satisfy the following condition for the prescribed small constant ϵ

$\{|\Omega_i - \omega_i| \leq \epsilon \quad (\text{for } i = 1, 2, \cdots, N + M + 2)\}$

End.

IV. Filter Properties

In this section, we will discuss how to select the extremal frequencies and describe some properties of lowpass filters, bandpass filters, and so on.

A. Lowpass and Highpass Filters

First, we consider the design of lowpass filters. By selecting extremal frequencies $\omega_{ip}(\omega_p = \omega_{0p} > \omega_{1p} > \cdots > \omega_{mp} \geq 0), \omega_{is}(\omega_s = \omega_{0s} < \omega_{1s} < \cdots < \omega_{ns} \leq \pi)$ in the passband $[0, \omega_p]$ and stopband $[\omega_s, \pi]$, respectively, we formulate $|H(e^{j\omega_{ip}})|^2 = 1$ for $(i = 1, 3, \cdots)$ and $|H(e^{j\omega_{ip}})|^2 = 1 - \delta_p$ for $(i = 0, 2, \cdots)$ in the passband and $|H(e^{j\omega_{is}})|^2 = \delta_s$ for $(i = 0, 2, \cdots)$ and $|H(e^{j\omega_{is}})|^2 = 0$ for $(i = 1, 3, \cdots)$ in the stopband, where $\delta_p(>0)$ and $\delta_s(>0)$ are the passband and stopband error. Now, we discuss the number of the extremal frequencies in the passband and stopband. The number of the unknown coefficients in (3) are $(N + M + 1)$ due to $d_0 = 1$. Hence, the number m and n must be set to satisfy

$$m + n = M + N. \tag{15}$$

We know that the extremal frequencies $\omega_{is}(i = 1, 3, \cdots)$ are zeros of $H(z)$. From (5), we see that the extremal frequencies $\omega_{ip}(i = 1, 3, \cdots)$ are zeros of $G(z)$. $H(z)$ has at most $[\frac{N+1}{2}]$ zeros on the upper unit semicircle, and $G(z)$ has at most $[\frac{L+1}{2}]$ zeros, where $[*]$ indicates the integral part of $*$. Hence, the numbers m and n are limited in

$$\begin{cases} m \leq L \\ n \leq N. \end{cases} \tag{16}$$

When $N \leq M$, we have $L = M$. Then, from (15) and (16), the numbers m and n must be

$$\begin{cases} m = M \\ n = N. \end{cases} \tag{17}$$

In other words, all zeros of $H(z)$ and $G(z)$ are on the unit circle. Therefore, when $N \leq M$, the optimal filter $H(z)$ has all zeros on the unit circle, and there is no extra ripple filter. If $N < M$ and N is even, the extremal frequency may not arise at $\omega = \pi$, in particular, in the case with wide stopband and $N \ll M$ (see [12], [15]).

We rewrite (3) as

$$|H(e^{j\omega})|^2 = \frac{1}{1 + K(\omega)} \tag{18}$$

$$K(\omega) = \frac{d_0 - c_0 + 2\sum_{i=1}^{L}(d_i - c_i)\cos i\omega}{c_0 + 2\sum_{i=1}^{N} c_i \cos i\omega}. \tag{19}$$

When $N \leq M$, let $\tilde{c}_i = kc_i$ and $\tilde{d}_i = d_i + (k - 1)c_i$ $(i = 0, 1, \cdots, N)$. $\tilde{d}_i = d_i(i = N + 1, \cdots, M)$ be another set of filter coefficients; then, $\tilde{K}(\omega) = K(\omega)/k$. In the passband, $K(\omega) \to 0$; then $\tilde{\delta}_p \simeq \delta_p/k$. In the stopband, $1/K(\omega) \to 0$; then, $\tilde{\delta}_s \simeq k\delta_s$. Therefore, we can adjust the passband and stopband error by only one parameter k while all zeros of $H(z)$ and $G(z)$ are kept fixed.

When $N > M$ and $L = N$, we have

$$\begin{cases} M \leq m \leq N \\ M \leq n \leq N. \end{cases} \tag{20}$$

In other words, $H(z)$ has at least M zeros on the unit circle. Let $\bar{H}(x) = |H(e^{j\omega})|^2_{\omega=\cos^{-1} x}$, where $\bar{H}(x)$ can have at most $(M + N - 1)$ derivatives equal to 0. Hence the number of extra ripple is at most one in the case of both lowpass and highpass filters. When M is even, there exist $(N - M)$ extra ripple filters. When M is odd, it is proved in [12] that there is no extra ripple filter with all zeros on the unit circle. When $H(z)$ has some zeros off the unit circle, we have found in the practical designs that it is also impossible to obtain the extra ripple filter (see Example 3). The same properties can be obtained in the case of highpass filters.

B. Bandpass Filters and Bandstop Filters

We consider the design of bandpass and bandstop filters. We select $(m + 1)$ and $(n + 1)$ extremal frequencies in the passband(s) and stopband(s), respectively. In the case of bandpass filters, the number m must be even. By the number of zeros of $H(z)$ and $G(z)$, the number m and n are limited in

$$\begin{cases} m \leq L \\ n \leq N + 1. \end{cases} \tag{21}$$

In other words, there exists at least one extra ripple filter regardless of M and N. In the practical computation, we cannot set $n = N + 1$ because there is an extra ripple that cannot be taken into account in the stopbands if $n = N + 1$. When $N \leq M$, m and n must be

$$\begin{cases} m = M \\ n = N. \end{cases} \tag{22}$$

TABLE I
NUMBER OF EXTREMAL FREQUENCIES AND ZEROS OFF THE UNIT CIRCLE FOR LOWPASS FILTERS WITH $N = 6, M = 4, \omega_p = 0.4\pi, \omega_s = 0.5\pi$

Weighting function in stopband ($W_s(\omega)$)	Number of extremal frequencies	Number of zeros off the unit circle
$W_s(\omega) \geq 70$	12	0
$70 > W_s(\omega) > 21.2$	12 ($\omega_{mp} \neq 0$)	0
$W_s(\omega) = 21.2$	13 (extra ripple)	0
$21.2 > W_s(\omega) > 10$	12 ($\omega_{ns} \neq \pi$)	0
$W_s(\omega) = 10$	12	0 (double zeros at $z = -1$)
$10 > W_s(\omega) > 0.21$	12	1 ($z_{0r} < 0$)
$W_s(\omega) = 0.21$	12	0 ($c_N = 0$)
$0.21 > W_s(\omega) \geq 0.0031$	12	1 ($z_{0r} > 0$)
$0.0031 > W_s(\omega) > 0.0016$	12 ($\omega_{mp} \neq 0$)	1 ($z_{0r} > 0$)
$W_s(\omega) = 0.0016$	13 (extra ripple)	1 ($z_{0r} > 0$)
$0.0016 > W_s(\omega) > 0.0003$	12 ($\omega_{ns} \neq \pi$)	2 ($z_{0r1} < 0, z_{0r2} > 0$)
$0.0003 \geq W_s(\omega)$	12	2 ($z_{0r1} < 0, z_{0r2} > 0$)

Therefore, the denominator of order M has to be even, $H(z)$ has at most one extra ripple in the stopbands, and $G(z)$ has all zeros on the unit circle. When $N > M$, we can get

$$\begin{cases} M \leq m \leq N \\ M \leq n \leq N \end{cases} \tag{23}$$

and the number of extra ripples is at most three.

In the case of stopband filters, the number n must be even. Similarly, m and n are limited in

$$\begin{cases} m \leq L+1 \\ n \leq N \end{cases}. \tag{24}$$

When $N \leq M$, m and n must satisfy (22). Hence, the numerator order N has to be even, and $H(z)$ has all zeros on the unit circle and at most one extra ripple in the passbands. When $N > M$, m and n must satisfy (23).

V. Design Examples

In this section, we present several design examples to demonstrate the effectiveness of the proposed method.

Example 1: We consider design of a lowpass filter with the following specifications: $N = 6$, $M = 4$, and $\omega_p = 0.4\pi, \omega_s = 0.5\pi$. By setting passband weighting function $W_p(\omega) = 1$, we have designed many filters with various stopband weighting functions $W_s(\omega)$. The number and locations of extremal frequencies and zeros off the unit circle of the resulting filters are shown in Table I. We see from Table I that the filters with $W_s(\omega) \geq 10$ have all zeros on the unit circle. In particular, there exist double zeros at $z = -1$ when $W_s(\omega) = 10$. The filters have one real zero z_{0r} in the interval $(-1, 0)$ of the z plane when $10 > W_s(\omega) > 0.21$ and one real zero z_{0r} in the interval $(0, 1)$ when $0.21 > W_s(\omega) \geq 0.0016$. When $W_s(\omega) = 0.21$, the real zeros cancels each other out with the pole at the origin due to $c_N = 0$; hence, the filter degenerates into the extra ripple filter with $N = 5$ and $M = 4$ that has all zeros on the unit circle. The filters with $W_s(\omega) < 0.0016$ have two real zeros z_{0r1} and z_{0r2} lying in the intervals $(-1, 0)$ and $(0, 1)$, respectively. We see also that there are two extra ripple filters with $W_s(\omega) = 21.2$ and $W_s(\omega) = 0.0016$, which can be considered to be the degenerate situations of the filter with $N = 6$ and $M = 5$ when $d_M = 0$ or $N = 7$ and $M = 4$ when $c_N = 0$. The extremal frequency ω_{mp} does not arise at $\omega = 0$ when $70 > W_s(\omega) > 21.2$ and $0.0031 > W_s(\omega) > 0.0016$, and ω_{ns} does not arise at $\omega = \pi$ when $21.2 > W_s(\omega) > 10$ and $0.0016 > W_s(\omega) > 0.0003$, as shown in Fig. 2. It is clear from Fig. 2 that the extra ripple filter with $W_s(\omega) = 21.2$ is not the optimum filter with all zeros on the unit circle and the minimum passband ripple. The proof of [12] is wrong when N is even. Let $\tilde{H}_1(\omega) = |H_1(e^{j\omega})|^2$ be the extra ripple filter with all zeros on the unit circle, let $\tilde{H}_2(\omega) = |H_2(e^{j\omega})|^2$ be an optimum filter with all zeros on the unit circle and the smaller passband ripple, and let $\hat{H}(\omega) = \tilde{H}_1(\omega) - \tilde{H}_2(\omega)$. In [12], it is proved that $\hat{H}(\omega)$ must have $(M + 1)$ zeros in the passband and N zeros in the stopband if $\tilde{H}_2(\omega)$ exists. Hence, it is impossible because $\hat{H}(\omega)$ has at most $(M + N)$ zeros. However, it is seen from Fig. 2 that $\hat{H}(\omega)$ has only $(N - 1)$ zeros in the stopband due to $\omega_{ns} \neq \pi$. Therefore, we can conclude that the optimum filter with the minimum passband ripple and all zeros on the unit circle is the filter with double zeros at $z = -1$ but not the extra ripple filter when N is even. Since the extra ripple will arise when $N > M$ and M is even, the initial number n and m can be arbitrarily set between M and N; then, n and m are automatically adjusted at each iteration. The procedures of [12] and [13] can design only the filters with $W_s(\omega) \geq 21.2$. The procedure of [15] improves the algorithm of [12], and then, we can design the filters with $0.21 \geq W_s(\omega) \geq 0.0031$ as well.

Example 2: We consider design of the lowpass filter of [15] with $N = 16$, $M = 2$, and $\omega_p = 0.2\pi, \omega_s = 0.3\pi$ for comparison purposes. The passband weighting function is set to $W_p(\omega) = 1$. First, we designed the filter with the stopband weighting function $W_s(\omega) = 2.77 \times 10^5$. The magnitude response of the filter is shown in the solid line in Fig. 3, and

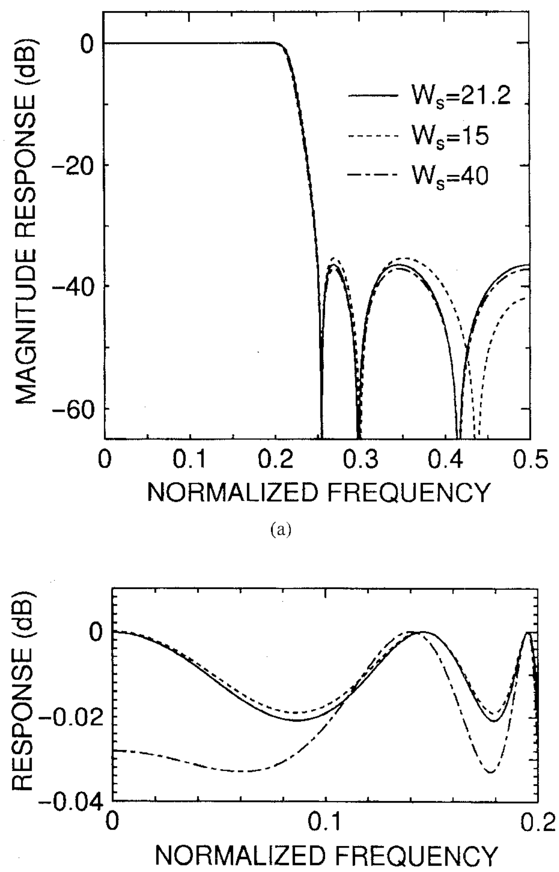


Fig. 2. Magnitude responses of Example 1: (a) Log magnitude in decibels; (b) passband detail.

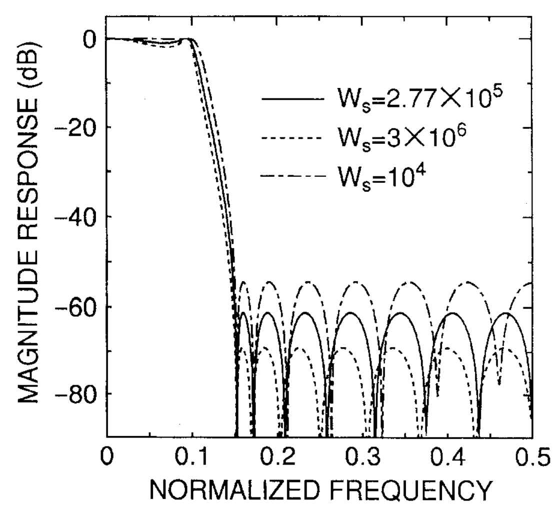


Fig. 3. Magnitude responses of Example 2.

the passband and stopband attenuation are 1 and 61.3 dB, respectively. The resulting filter is the same as that of [15], and one real zero lies in the interval $(0, 1)$ of the z plane. We have also designed two filters with $W_s(\omega) = 3 \times 10^6$

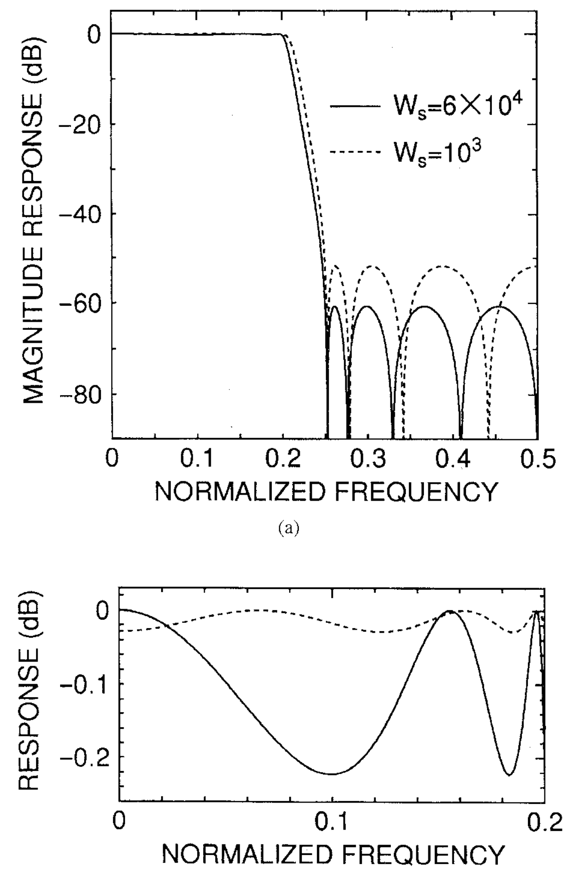
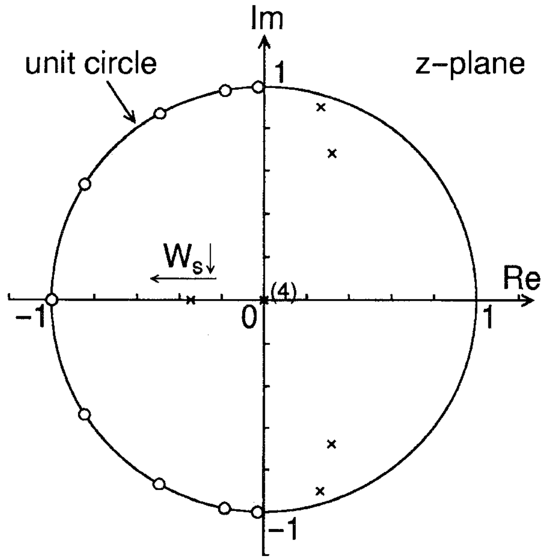


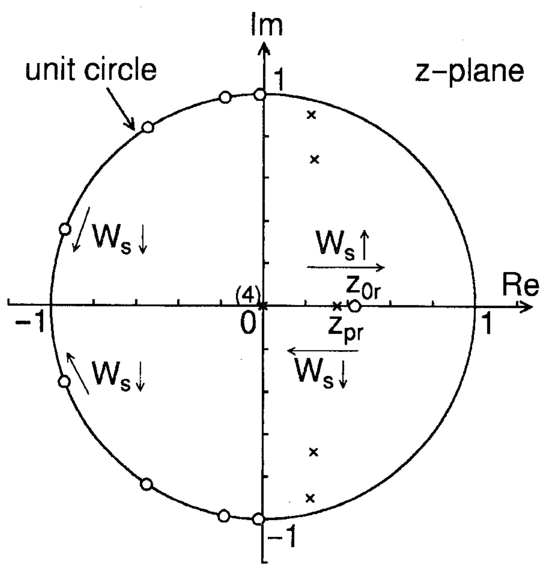
Fig. 4. Magnitude responses of Example 3: (a) Log magnitude in decibels; (b) passband detail.

and $W_s(\omega) = 10^4$, and the magnitude responses are shown in Fig. 3. The filter with $W_s(\omega) = 3 \times 10^6$ has one real zero in the interval $(-1, 0)$, and one of $W_s(\omega) = 10^4$ has two real zeros, which cannot be designed by the procedure of [15].

Example 3: We consider design of a lowpass filter with $N = 9$, $M = 5$, and $\omega_p = 0.4\pi, \omega_s = 0.5\pi$. The passband weighting function is set to $W_p(\omega) = 1$. First, we set the initial number $n = N$ and $m = M$ and designed the filter with the stopband weighting function $W_s(\omega) = 6 \times 10^4$. The magnitude response is shown in the solid line in Fig. 4, and the resulting filter has all zeros on the unit circle and one real pole in the interval $(-1, 0)$ of the z plane as shown in Fig. 5. We decreased the stopband weighting function $W_s(\omega)$ to design the filter. We found that the real pole moves forward to $z = -1$ with a decreasing $W_s(\omega)$ and ultimately cancels with the zero at $z = -1$, but it cannot produce an extra ripple. In order to design the filter with smaller $W_s(\omega)$, we have to reset the initial number $n = N - 1$ and $m = M + 1$. The filter with $W_s(\omega) = 10^3$ has been designed, and the magnitude response is shown in the dashed line in Fig. 4. The filter has one real zero z_{0r} in the interval $(0, 1)$ and one real pole z_{pr} that lies in the left of the real zero (i.e., $z_{pr} < z_{0r}$) as shown Fig. 5. When

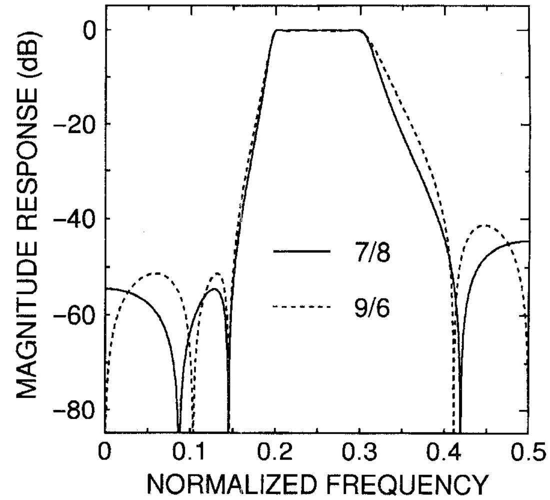


(a)

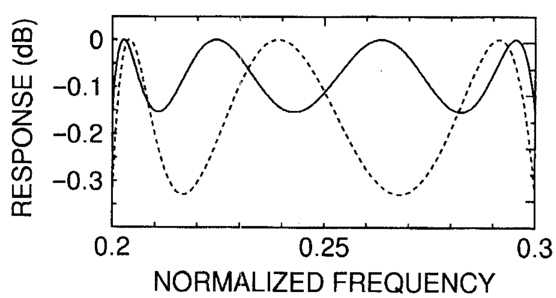


(b)

Fig. 5. Pole-zero location: (a) $W_s = 6 \times 10^4$; (b) $W_s = 10^3$.



(a)

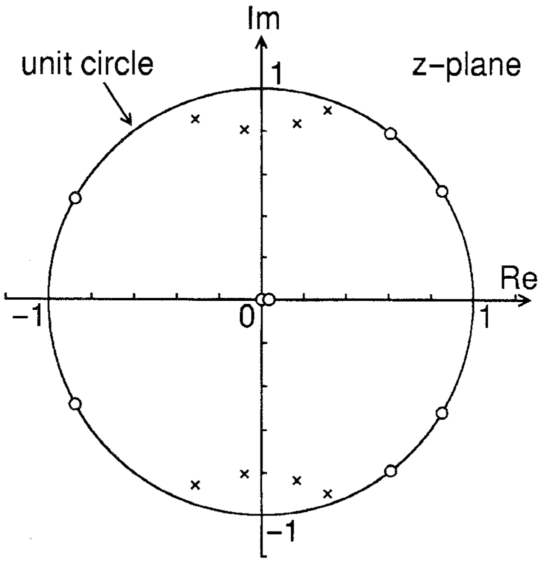


(b)

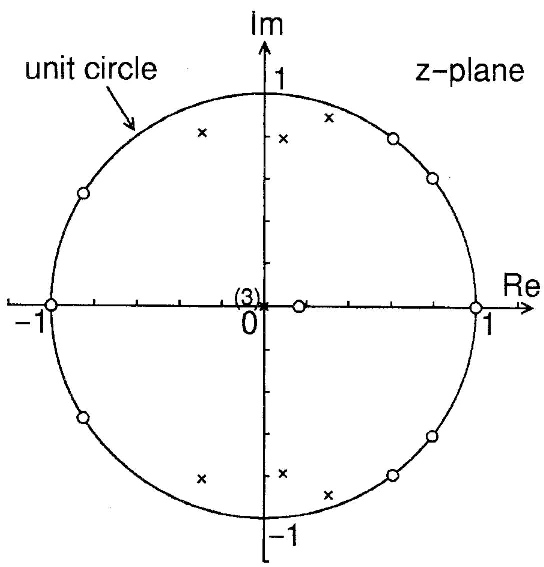
Fig. 6. Magnitude responses of Example 4. (a) Log magnitude in decibels; (b) passband detail.

$W_s(\omega)$ increases, the real pole and real zero move forward to $z = 1$ and ultimately cancel each other at $z = 1$. When $W_s(\omega)$ decreases, the real pole moves forward to $z = -1$ and ultimately cancels with the zero at $z = -1$, whereas the real zero is kept in the interval $(0, 1)$. However, the extra ripple cannot also arise. In other cases, we have observed similar situations. Therefore, we can conclude that there is no extra ripple filter when M is odd, and the initial number n and m must be set according to the weighting function. It can be explained that if there is an extra ripple filter when M is odd, it must be the degenerate situation of the filter with the numerator order $N + 1$ and the denominator order M when $c_{N+1} = 0$ or the numerator order N and the denominator order $M+1$ when $d_{M+1} = 0$. It is seen in the above examples that the degeneration happens when the real zero or real pole moves from the interval $(-1, 0)$ to $(0, 1)$ with a decreasing or increasing $W_s(\omega)$ and is located just at the origin. When the denominator order M is odd, we have found that there is at most one real zero inside the unit circle, which is kept in the interval $(0, 1)$ regardless of $W_s(\omega)$. Hence, the filter with the numerator order $N + 1$ and the denominator order M cannot have one zero at the origin to produce $c_{N+1} = 0$. In the case of the filter with the numerator order N and the denominator order $M + 1$, there are $M + 1$ complex poles because the denominator order $M + 1$ is even. The complex poles cannot move to the origin to produce $d_{M+1} = 0$. Therefore, there is no extra ripple filter when M is odd.

Example 4: We consider the design of bandpass filters with $N + M = 15$, and the weighting function $(10^4, 1, 10^3)$ in first stopband $[0, 0.3\pi]$, passband $[0.4\pi, 0.6\pi]$, and second stopband $[0.8\pi, \pi]$, respectively. Two filters with $N = 7$ and $M = 8$ or $N = 9$ and $M = 6$ are designed, and the magnitude responses are shown in Fig. 6. The first stopband attenuation of two filters is 54.58 and 51.36 dB respectively, and there is difference of 3.22 dB. Their pole-zero diagrams are shown in Fig. 7. It is seen in Fig. 7 that one real zero of the filter with



(a)



(b)

Fig. 7. Pole-zero location: (a) $N = 7, M = 8$; (b) $N = 9, M = 6$.

$N = 7$ and $M = 8$ is located near the origin. Although this real zero hardly ever contributes to the magnitude response, the filter of $N = 7$ and $M = 8$ is better than one of $N = 9$ and $M = 6$ because it has more poles off the origin. Therefore, the filters with $N < M$ are effective in the case with a narrow passband. In the case with a wide passband, we have also observed that the filters with $N > M$ are effective. The conclusion is the same with lowpass filters.

VI. Conclusion

In this paper, we have proposed a new method for designing IIR digital filters with optimum magnitude response in the Chebyshev sense and different order numerator and denominator. The design procedure is based on the formulation of a generalized eigenvalue problem using a Remez exchange algorithm. We have introduced a new and very simple selection rule where the rational interpolation is performed if and only if the positive minimum eigenvalue is chosen. Therefore, the solution of the rational interpolation problem can be obtained by computing only one eigenvector corresponding to the positive minimum eigenvalue, and the optimal filter coefficients are easily obtained through a few iterations. The design algorithm not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the positive minimum eigenvalue. The proposed method can be extended to the design of filters with arbitrary magnitude response specifications and multiband.

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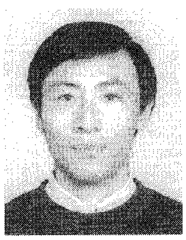
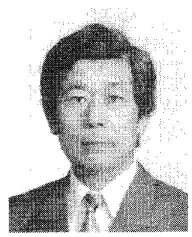


Xi Zhang (M'94) was born in Jiangsu, China, on December 23, 1963. He received the B.E. degree in electrical engineering from the Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 1984 and the M.E. and Ph.D. degrees in communication and system engineering from the University of Electro-Communications (UEC), Tokyo, Japan, in 1990 and 1993, respectively.

From 1984 to 1987, he was with the Department of Electrical Engineering at NUAA as a research assistant. Currently, he is with the Department of Communications and Systems at UEC as a research assistant. He was a recipient of the Award of Science and Technology Progress of China in 1987. His research interests are in the areas of digital signal processing, filter design, wavelets, and neural networks.

Dr. Zhang is a member of IEICE of Japan.



Hiroshi Iwakura (M'78) was born in Nagano, Japan, on September 3, 1939. He graduated from the Faculty of Electro-Communications, the University of Electro-Communications, Tokyo, Japan, in 1963. He received the M.E. degree in electrical engineering from Tokyo Metropolitan University, Tokyo, Japan, in 1968 and the Ph.D. degree in electronic engineering from Tokyo Institute of Technology, Tokyo, Japan, in 1990, respectively.

In 1968, he joined the faculty of Electro-Communications of the University of Electro-Communications (UEC), and since then, as a faculty member, he has engaged in education and research in the fields of microwave theory, circuit and system theory, digital signal processing, and others. He is now a Professor of the Department of Communications and Systems at UEC.

Prof. Iwakura is a member of IEICE of Japan.