Note that popular adaptive algorithms can be naturally extended to the FBT case. Those include single- or double-tree algorithms [3], which can be described using a concept of tree-structured basis libraries. Although we mainly considered audio signals, the proposed framework, in its generality and the good results, suggest the applications of FBT methods in other fields of signal processing.

Further research on FBT's and M-band wavelet packets is conducted with specific interest in real-time signal compression and the construction of specific application-adapted FBT's. The incorporation of more flexible psychoacoustic models [18] also seems promising in denoising and compression applications and is currently being investigated.

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Design of IIR Digital Allpass Filters Based on Eigenvalue Problem

Xi Zhang and Hiroshi Iwakura

***Abstract*—A new method is proposed for designing IIR digital allpass filters with an equiripple phase response that can be proven to be optimal in the Chebyshev sense. The proposed procedure is based on the formulation of an eigenvalue problem by using the Remez exchange algorithm. Since there exists more than one eigenvalue in the general eigenvalue problem, we introduce a new and very simple selection rule for the eigenvalue to be searched for, where the rational interpolation is performed if and only if the real maximum eigenvalue is chosen. Therefore, the solution of the rational interpolation problem can be gotten by computing only one eigenvector corresponding to the real maximum eigenvalue, and the optimal filter coefficients are easily obtained through a few iterations without any initial guess of the solution. The design algorithm proposed in this correspondence not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the real maximum eigenvalue. Two examples are designed to demonstrate the effectiveness of this method.**

***Index Terms*— Digital allpass filter, eigenvalue problem, Remez exchange algorithm.**

I. Introduction

In many signal processing applications, it is necessary to design an allpass filter whose phase approximates the specified phase response in the Chebyshev sense. IIR digital allpass filters possess unit magnitude response at all frequencies and are a basic scalar lossless building block. One of the most widely used applications of allpass filters is phase or group delay equalizers. In recent years, the interconnections of allpass filters have found numerous applications in many practical filtering problems such as low-sensitivity filter structures, complementary filter banks, multirate filtering, and so on [1]–[19]. Although it has been shown in [2] that the equiripple solutions to group delay approximation are not necessarily the optimal minimax (Chebyshev) solution, we can prove in this correspondence that the equiripple phase approximations are optimal in the Chebyshev sense. In this correspondence, we present a new method for designing IIR digital allpass filters with an equiripple phase response based on the eigenvalue problem.

The problem of designing allpass filters to approximate the specified phase response in the Chebyshev sense has no solutions in an explicit form. Several methods have been proposed based on the minimum p-error criterion approximation [1], a generalized exchange method [2], and linear programming algorithm [3]. However, the disadvantages of these approaches include the need of strict initial conditions and/or heavy computational burden. In [4], a method of iteratively linearizing the nonlinear constraints required in the nonlinear programming problem and applying the Remez exchange algorithm to the amplitude error function between the desired and designed frequency responses is presented, but it is not guaranteed that the convergent solution can be obtained. In [6]–[9], the weighted

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least squares (WLS) algorithms are used to design allpass filters. However, the WLS algorithms must employ an iterative reweighted procedure to obtain an equiripple phase response. There do not yet exist any well-established easy-to-use approaches for IIR digital allpass filter design.

The purpose of this correspondence is to develop a new design method based on the eigenvalue problem for IIR digital allpass filters with an equiripple phase response. By using the Remez exchange algorithm, we formulate the design problem in the form of an eigenvalue problem. The solution of the rational interpolation problem can be obtained by solving the eigenvalue problem, which was already proposed by Werner [21]. There exist more than one eigenvalue in the general eigenvalue problem; then, we must search for one eigenvalue that corresponds with the solution of the rational interpolation problem. However, Werner did not give a selection rule for the eigenvalue to be searched for. In this correspondence, we introduce a new and very simple selection rule where the rational interpolation is performed if and only if the real maximum eigenvalue is chosen. Therefore, we can obtain the solution by computing only one eigenvector corresponding to the real maximum eigenvalue. In order to obtain an equiripple phase response, we make use of an iteration procedure so that the optimal filter coefficients can be easily obtained. The new algorithm proposed in this paper not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the real maximum eigenvalue. In general, the design algorithm converges rapidly with a few iterations and computes efficiently without any initial guess of the solution. Two examples are designed to demonstrate the effectiveness of this method.

II. Definitions and Properties

The transfer function of an Nth-order IIR allpass filter is defined as [1]–[9]

$$A_N(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n^* z^{-n}} \tag{1}$$

where $a_n = a_{nr} + j a_{ni}$ are complex coefficients in general, and a_n^* denotes the complex conjugate of a_n. When $a_{ni} = 0$, i.e., a_n are real, $A_N(z)$ is a real allpass filter, which is a special case of complex allpass filters. All poles and zeros of $A_N(z)$ occur in mirror-image pairs, and the frequency response $A_N(e^{j\omega})$ exhibits unit magnitude at all frequencies, i.e., $|A_N(e^{j\omega})| \equiv 1$ for all ω. The phase response of $A_N(z)$ is given by

$$\theta(\omega) = -N\omega + 2\tan^{-1} \frac{\sum_{n=0}^{N} \{a_{nr} \sin n\omega + a_{ni} \cos n\omega\}}{\sum_{n=0}^{N} \{a_{nr} \cos n\omega - a_{ni} \sin n\omega\}}. \tag{2}$$

If all poles locate inside the unit circle, then $A_N(z)$ is stable. The phase response decreases monotonically with an increasing frequency, and $\theta(\pi) = \theta(-\pi) - 2N\pi$ [4]. When one pole locates at the origin, it is seen that $A_N(z) = z^{-1}A_{N-1}(z)$ due to $a_N = 0$. Then, the delay z^{-N} is a special case of $A_N(z)$ if all poles locate at the origin. When k poles locate outside the unit circle, we can divide $A_N(z)$ into two stable allpass filters $A_{N-k}(z)$ and $A_k(z)$, i.e.,

$$A_N(z) = \frac{A_{N-k}(z)}{A_k(z)}. \tag{3}$$

The phase response $\theta(\omega)$ of $A_N(z)$ is the phase difference between $A_{N-k}(z)$ and $A_k(z)$, and $\theta(\pi) = \theta(-\pi) - 2(N-2k)\pi$. Hence, the desired phase response is required to satisfy the above properties and cannot be arbitrarily specified.

Let $\theta_d(\omega)$ be the desired phase response; the difference $\theta_e(\omega)$ between $\theta(\omega)$ and $\theta_d(\omega)$ is

$$\begin{aligned} \exp\{j\theta_e(\omega)\} &= \exp\{j(\theta(\omega) - \theta_d(\omega))\} \\ &= \frac{\sum_{n=0}^{N} a_n \exp\left\{j\left(n\omega - \frac{N\omega + \theta_d(\omega)}{2}\right)\right\}}{\sum_{n=0}^{N} a_n^* \exp\left\{-j\left(n\omega - \frac{N\omega + \theta_d(\omega)}{2}\right)\right\}} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \theta_e(\omega) &= 2\tan^{-1} \frac{\sum_{n=0}^{N} \{a_{nr} \sin \Theta_n(\omega) + a_{ni} \cos \Theta_n(\omega)\}}{\sum_{n=0}^{N} \{a_{nr} \cos \Theta_n(\omega) - a_{ni} \sin \Theta_n(\omega)\}} \\ &= 2\tan^{-1} \Phi(\omega) \end{aligned} \tag{5}$$

where $\Theta_n(\omega) = (n - N/2)\omega - (\theta_d(\omega)/2)$. The phase Chebyshev approximation minimizes the maximum-phase error of $\theta_e(\omega)$ in the interest band(s). It has been shown in [2] that the equiripple solution to group delay approximation is not necessarily the optimal Chebyshev solution. In the following, we will prove that the equiripple phase response with at least $2(N+1)$ extremal points is optimal in the Chebyshev sense.

Theorem I: The equiripple phase response with at least $2(N+1)$ extremal points is the optimal Chebyshev solution.

Proof: Let a_n^e be the equiripple solution and a_n^o the optimal Chebyshev solution. $\Phi_e(\omega)$ and $\Phi_o(\omega)$ are their error functions, respectively. Since the equiripple solution has at least $2(N+1)$ extremal points, there exist at least $2N+1$ frequency points $\bar{\omega}_i$ that satisfy

$$\Phi_e(\bar{\omega}_i) = \Phi_o(\bar{\omega}_i). \tag{6}$$

Substituting (5) into (6), we can obtain

$$\begin{aligned} &\sum_{n=0}^{N} \{a_{nr}^e \sin \Theta_n(\bar{\omega}_i) + a_{ni}^e \cos \Theta_n(\bar{\omega}_i)\} \\ &\quad \cdot \sum_{m=0}^{N} \{a_{mr}^o \cos \Theta_m(\bar{\omega}_i) - a_{mi}^o \sin \Theta_m(\bar{\omega}_i)\} \\ &\quad - \sum_{n=0}^{N} \{a_{nr}^e \cos \Theta_n(\bar{\omega}_i) - a_{ni}^e \sin \Theta_n(\bar{\omega}_i)\} \\ &\quad \cdot \sum_{m=0}^{N} \{a_{mr}^o \sin \Theta_m(\bar{\omega}_i) + a_{mi}^o \cos \Theta_m(\bar{\omega}_i)\} \\ &= \sum_{n=0}^{N} \sum_{m=0}^{N} (a_{nr}^e a_{mr}^o + a_{ni}^e a_{mi}^o)\{\sin \Theta_n(\bar{\omega}_i) \cos \Theta_m(\bar{\omega}_i) \\ &\quad - \cos \Theta_n(\bar{\omega}_i) \sin \Theta_m(\bar{\omega}_i)\} \\ &\quad + \sum_{n=0}^{N} \sum_{m=0}^{N} (a_{ni}^e a_{mr}^o - a_{nr}^e a_{mi}^o)\{\cos \Theta_n(\bar{\omega}_i) \cos \Theta_m(\bar{\omega}_i) \\ &\quad + \sin \Theta_n(\bar{\omega}_i) \sin \Theta_m(\bar{\omega}_i)\} \\ &= \sum_{n=0}^{N} \sum_{m=0}^{N} (a_{nr}^e a_{mr}^o + a_{ni}^e a_{mi}^o) \sin(n-m)\bar{\omega}_i \\ &\quad + \sum_{n=0}^{N} \sum_{m=0}^{N} (a_{ni}^e a_{mr}^o - a_{nr}^e a_{mi}^o) \cos(n-m)\bar{\omega}_i \\ &= \sum_{n=1}^{N} b_n \sin n\bar{\omega}_i + \sum_{n=0}^{N} c_n \cos n\bar{\omega}_i = 0 \end{aligned} \tag{7}$$

where

$$\begin{cases} b_n = \sum_{m=0}^{N-n} \big(a^e_{(m+n)r}a^o_{mr} - a^e_{mr}a^o_{(m+n)r} \\ \quad +a^e_{(m+n)i}a^o_{mi} - a^e_{mi}a^o_{(m+n)i}\big) \\ c_n = \sum_{m=0}^{N-n} \big(a^e_{(m+n)i}a^o_{mr} + a^e_{mi}a^o_{(m+n)r} \\ \quad -a^e_{(m+n)r}a^o_{mi} - a^e_{mr}a^o_{(m+n)i}\big). \end{cases} \tag{8}$$

It is clear that (7) has at most $2N$ roots. To satisfy (6) at $2N+1$ frequency points $\bar{\omega}_i$, all b_n and c_n must be equal to 0. Therefore, from (8), we get

$$a^e_n = Ca^o_n \qquad (\text{for all } n) \tag{9}$$

where C is a constant. We can conclude that the equiripple solution is the optimal Chebyshev solution. The theorem is proven.

III. Formulation Based on Eigenvalue Problem

In this section, we describe design of IIR allpass filters based on the eigenvalue problem. When the desired phase response $\theta_d(\omega)$ is specified in the interest band(s) $R \in (-\pi, \pi]$ and the filter order N is reasonably selected, the aim is to find a set of filter coefficients a_n to minimize the maximum error of $\theta_e(\omega)$. To solve the phase Chebyshev approximation problem, we use the Remez exchange algorithm and formulate the condition for $\Phi(\omega)$ of (5) in the form of an eigenvalue problem. We select $2(N+1)$ extremal frequencies ω_i $(i = 0, 1, \cdots, 2N+1)$ in the band(s) R and formulate $\Phi(\omega)$ as

$$W(\omega_i)\Phi(\omega_i) = W(\omega_i)\tan\frac{\theta_e(\omega_i)}{2} = (-1)^{(i+l)}\delta \tag{10}$$

where $W(\omega)$ is a weighting function, and $l = 0$ or 1 to guarantee $\delta > 0$. The denominator polynomial of $\Phi(\omega)$ must satisfy

$$\sum_{n=0}^{N}\{a_{nr}\cos\Theta_n(\omega) - a_{ni}\sin\Theta_n(\omega)\} \neq 0 \qquad (\omega \in R). \tag{11}$$

Substituting (5) into (10), we can rewrite (10) in matrix form as

$$\boldsymbol{PA} = \delta\boldsymbol{QA} \tag{12}$$

where $\boldsymbol{A} = [a_{0r}, a_{1r}, \cdots, a_{Nr}, a_{0i}, a_{1i}, \cdots, a_{Ni}]^T$, and the elements of the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are given by

$$P_{ij} = \begin{cases} W(\omega_i)\sin\Theta_j(\omega_i), & (0 \le j \le N) \\ W(\omega_i)\cos\Theta_{(j-N-1)}(\omega_i), & (N < j \le 2N+1) \end{cases} \tag{13}$$

$$Q_{ij} = \begin{cases} (-1)^{(i+l)}\cos\Theta_j(\omega_i), & (0 \le j \le N) \\ (-1)^{(i+l+1)}\sin\Theta_{(j-N-1)}(\omega_i), & (N < j \le 2N+1). \end{cases} \tag{14}$$

Once $\theta_d(\omega)$ and $W(\omega)$ are given, it is seen from (13) and (14) that the elements of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are known. Therefore, it should be noted that (12) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue, and $\boldsymbol{A}$ is a corresponding eigenvector. It is well known that there is a nontrivial solution $\boldsymbol{A}$ in (12) if and only if the determinant satisfies

$$|\boldsymbol{P} - \delta\boldsymbol{Q}| = 0. \tag{15}$$

Since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $2(N+1) \times 2(N+1)$ matrices, (15) has more than one solution of δ in general. We can obtain at least two solutions by solving the eigenvalue problem of (12). To minimize the maximum-phase error, the filter coefficients must satisfy the condition of (11). However, it is not guaranteed that the solutions obtained from (12) have satisfied (11). Therefore, we must search for the solution that satisfies (11) among the obtained solutions. Assuming that $\boldsymbol{P}$ is a singular matrix, we can get a solution by solving $\boldsymbol{PA} = \mathbf{o}$ that satisfies

$$\sum_{n=0}^{N}\{a_{nr}\sin\Theta_n(\omega) + a_{ni}\cos\Theta_n(\omega)\} = \sum_{n=0}^{N}\|a_n\|\sin(\Theta_n(\omega) + \varphi_n) = 0 \qquad (\omega \in R) \tag{16}$$

where $\|a_n\| = \sqrt{a_{nr}^2 + a_{ni}^2}$ and $\varphi_n = \tan^{-1}(a_{ni}/a_{nr})$. It is required from (16) that $a_n = 0$ or $\Theta_n(\omega) + \varphi_n = m\pi$ (m: integer) for all n. Since at least one of a_n is not equal to 0, we assume that $a_L \neq 0$; then, $\theta_d(\omega) = -(N-2L)\omega + 2\varphi_L - 2m\pi$. It is a pure delay $z^{-(N-2L)}$ plus a constant phase $2\varphi_L$. This case does not need to be considered in practical designs. Therefore, $\boldsymbol{P}$ is a nonsingular matrix in general. Equation (12) can be rewritten into the standard eigenvalue problem

$$\boldsymbol{TA} = \lambda\,\boldsymbol{A} \tag{17}$$

where $\boldsymbol{T} = \boldsymbol{P}^{-1}\boldsymbol{Q}$, and $\lambda = 1/\delta$. Here, will we ask whether (17) has a solution that satisfies (11). If the solution exists, which eigenvalue corresponds to the solution? We see from (10) that the sign change of $\Phi(\omega)$ is caused by the sign change of either the numerator or denominator polynomial. When the numerator polynomial changes its sign, $\Phi(\omega)$ crosses 0 to change its sign. When the denominator polynomial changes its sign, $\Phi(\omega)$ crosses ∞. Therefore, there exists more than one solution, depending on the sign change of $\Phi(\omega)$ through 0 or ∞. To satisfy (11), $\Phi(\omega)$ must change its sign through 0. When the optimum Chebyshev approximation to the desired response exists, there are $2(N+1)$ extremal frequencies of $\Phi(\omega)$ [2], [20]–[23]. Hence, (17) has at least one solution that satisfies (11) if the extremal frequencies are appropriately selected. By the uniqueness of the optimal solution, the solution is unique. Now, we answer the second question. In (10), we can choose $l = 0$ or 1 to guarantee that the solution that satisfies (11) and has a positive error δ. Therefore, we seek only the positive and real eigenvalues.

Theorem II: The real maximum eigenvalue corresponds to the solution that satisfies (11) when the optimum Chebyshev approximation exists.

Proof: Let a^o_n be the solution with $\delta_o\,(>0)$ that satisfies (11), and let a^n_n be another solution with $\delta_n\,(>0)$ that does not satisfy (11). $\Phi_o(\omega)$ and $\Phi_n(\omega)$ are their error functions, respectively, and $E(\omega) = \Phi_o(\omega) - \Phi_n(\omega)$.

A) Assume that $\delta_o = \delta_n$. We have $E(\omega_i) = 0$ from (10). There are $2(N+1)$ extremal frequencies ω_i within $(-\pi, \pi]$. However, we know from (7) that $E(\omega)$ has at most $2N$ zeros within $(-\pi, \pi]$; then, it is impossible to have $2(N+1)$ zeros. Therefore, we conclude that $\delta_o \neq \delta_n$.

B) Assume that $\delta_o > \delta_n$. It is seen in Fig. 1 that $E(\omega)$ has one zero in the interval $[\omega_i, \omega_{i+1}]$ when $\Phi_n(\omega)$ crosses 0 to change its sign and two zeros when $\Phi_n(\omega)$ crosses ∞. There are $2N+1$ interpolated intervals in $(-\pi, \pi]$. We assume that $\Phi_n(\omega)$ changes its sign through ∞ within I interpolated intervals; then, $E(\omega)$ has $2N+I+1$ zeros. However, it is impossible that $E(\omega)$ has more than $2N$ zeros. Therefore, we can conclude that $\delta_o < \delta_n$, and $\lambda_o = 1/\delta_o$ is the real maximum eigenvalue. The theorem is proven.

We have proven that the real maximum eigenvalue corresponds to the solution that satisfies (11). Therefore, we can obtain the solution of the rational interpolation problem by computing only one eigenvector corresponding to the real maximum eigenvalue without solving all

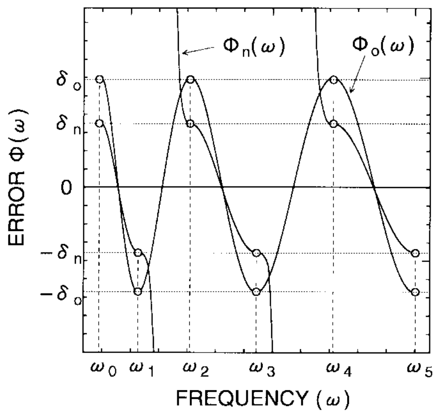

Fig. 1. Interpolation of $\Phi(\omega)$.

eigenvalues and eigenvectors. To obtain an equiripple phase response, we make use of an iteration procedure so that the optimal filter coefficients can be easily obtained. The design algorithm is shown as follows.

IV. Design Algorithm

Procedure {Design Algorithm of IIR Digital Allpass Filters}

Begin

1) Read N, $\theta_d(\omega)$, and $W(\omega)$.
2) Select the initial extremal frequencies Ω_i $(i = 0, 1, \cdots, 2N+1)$ in the band(s) R.

Repeat

3) Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, 2N+1$.
4) Compute $\boldsymbol{P}, \boldsymbol{Q}$ by using (13) and (14), and then, find the real maximum eigenvalue to obtain a set of filter coefficients a_n that satisfies (11).
5) Search the peak frequencies $\widehat{\omega}_i (i = 0, 1, \cdots, J)$ of $\Phi(\omega)$ within R.
6) Reject $(J - 2N - 2)$ superfluous peak frequencies, and store the remaining frequencies into the corresponding Ω_i.

Until

Satisfy the following condition for the prescribed small constant ϵ:

$$\sum_{n=0}^{2N+1} |\Omega_i - \omega_i| \le \epsilon$$

End.

In the above algorithm, a possible choice of the initial extremal frequencies is to pick these frequencies equally spaced in R. Other distributions may also be preferred to decrease the number of iterations. A major part of the computational time for our design method is spent in the computation of the eigenvector. Since we are interested in only one eigenvector corresponding to the real maximum eigenvalue, this computation can be done efficiently by using the iterative power method without invoking general methods such as the QR technique [24]. In each iteration, we have obtained the solution that satisfies (11) by computing the real maximum eigenvalue. We assume that the denominator polynomial of $\Phi(\omega)$ is positive without any loss in generality. Then, we can consider it to be a weighting function in the FIR applications. Therefore, the algorithm converges, in general, with a few iterations that are the same as the design of linear-phase FIR filters.

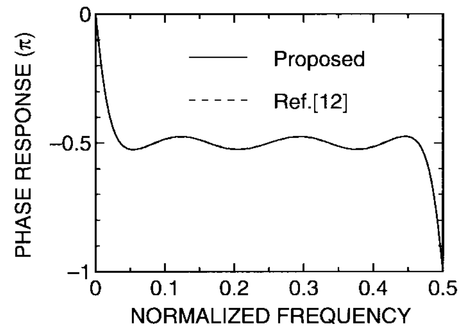

Fig. 2. Phase responses of Hilbert transformer.

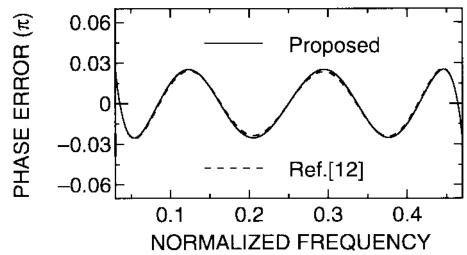

Fig. 3. Phase errors of Hilbert transformer.

V. Design Examples

A. Hilbert Transformers

Hilbert transform operations are useful in communication applications like modulation and demodulation [12]. The ideal frequency response of a discrete-time Hilbert transformer is given by

$$H_d(e^{j\omega}) = \begin{cases} j, & (-\pi < \omega < 0) \\ 0, & (\omega = 0 \text{ and } \pi) \\ -j, & (0 < \omega < \pi). \end{cases} \tag{18}$$

This ideal response can be approximated by using IIR allpass filters. The desired phase response of allpass filters is required to be

$$\theta_d(\omega) = -K\omega - \frac{\pi}{2} \qquad (\omega_l \le \omega \le \omega_u) \tag{19}$$

where K is the integer, and ω_l and ω_u are bandedge frequencies of the "care" band, respectively. It is the phase sum of the Hilbert transformer and a delay section z^{-K}. From the properties of allpass filters, the filter order must be chosen as $N_{\min} = K$ and $N_{\max} = K + 1$ to obtain stable allpass filters. Therefore, we can approximate this phase response by using the proposed method.

Example 1: We consider the design of the Hilbert transformer of [12] with $K = 5$, $\omega_l = 0.06\pi$, and $\omega_u = 0.94\pi$ for comparison purposes. The filter order is $N = 6$. The obtained phase response (without linear-phase $-K\omega$) and phase error are shown in the solid line in Figs. 2 and 3, respectively. It is seen that the phase response is equiripple. The result of [12] is shown in Figs. 2 and 3 as well. It is clear that both are almost same.

B. IIR Complimentary Filters with Approximately Linear Phase

It is known in [13] and [14] that the parallel interconnections of a delay section and an allpass filter can yield an approximately linear phase. However, this class of filters is restricted to have an integer delay. Here, we consider design of IIR filters with arbitrary delay, i.e., noninteger delay [19] using two allpass filters. IIR filters composed of two allpass filters are presented in [10]. Their transfer functions are

$$\begin{cases} H(z) = \frac{1}{2}\left[A_N(z) + A_M(z)\right] \\ G(z) = \frac{1}{2}\left[A_N(z) - A_M(z)\right] \end{cases} \tag{20}$$

where $A_N(z)$ and $A_M(z)$ are the Nth and Mth-order real allpass filters, respectively. Let $\theta_N(\omega)$, $\theta_M(\omega)$ be the phase responses of $A_N(z)$ and $A_M(z)$. The frequency responses of $H(z)$ and $G(z)$ are

$$\begin{cases} H(e^{j\omega}) = \exp\left[j\dfrac{\theta_N(\omega)+\theta_M(\omega)}{2}\right] \cos\dfrac{\theta_N(\omega)-\theta_M(\omega)}{2} \\ G(e^{j\omega}) = j\ \exp\left[j\dfrac{\theta_N(\omega)+\theta_M(\omega)}{2}\right] \sin\dfrac{\theta_N(\omega)-\theta_M(\omega)}{2}. \end{cases} \tag{21}$$

In the case of lowpass and highpass filter pairs, N and M must satisfy $N = M \pm 1$, and the phase responses of $A_N(z)$ and $A_M(z)$ must satisfy

$$\theta_N(\omega) - \theta_M(\omega) = \begin{cases} 0, & (0 \le \omega \le \omega_p) \\ \pm\pi, & (\omega_s \le \omega \le \pi). \end{cases} \tag{22}$$

The phase responses of $H(z)$ and $G(z)$ are required to be linear in both the passband and the stopband, i.e.,

$$\frac{\theta_N(\omega)+\theta_M(\omega)}{2} = \begin{cases} -\tau\omega, & (0 \le \omega \le \omega_p) \\ -\tau\omega + \epsilon, & (\omega_s \le \omega \le \pi) \end{cases} \tag{23}$$

where τ is group delay, and $\epsilon = (\tau - (N+M)/2)\pi$ since $\theta_N(\pi) = -N\pi$ and $\theta_M(\pi) = -M\pi$ for stable $A_N(z)$ and $A_M(z)$. Then, we can obtain the desired phase responses of $A_N(z)$ and $A_M(z)$ as

$$\theta_\Lambda^d(\omega) = \begin{cases} -(\Lambda + K_\Lambda)\omega, & (0 \le \omega \le \omega_p) \\ -(\Lambda + K_\Lambda)\omega + K_\Lambda\pi, & (\omega_s \le \omega \le \pi) \end{cases} (\Lambda = N, M) \tag{24}$$

where $K_\Lambda = \tau - \Lambda$. We have designed many allpass filters with various K_Λ and observed that only when $-2 \le K_\Lambda \le \Gamma_\Lambda$, the stable allpass filter can be obtained. When $K_\Lambda < -2$, the obtained solution is unstable, and when $K_\Lambda > \Gamma_\Lambda$, the allpass filter cannot be designed. Γ_Λ is decided by the order Λ and the width $(\omega_s - \omega_p)$ of the transition band. It is also because the phase response of the stable allpass filters must satisfy the property of monotonically decreasing in the transition band, i.e., $\theta_\Lambda(\omega_p) > \theta_\Lambda(\omega_s)$. Then, we have

$$\Gamma_\Lambda \approx \frac{\omega_s - \omega_p}{\pi - (\omega_s - \omega_p)}\Lambda. \tag{25}$$

Therefore, the group delay of $H(z)$ is restricted to $M_{\max} - 2 \le \tau \le M_{\min} + \Gamma_{M_{\min}}$, where $M_{\max} =$ Max$\{N, M\}$, and $M_{\min} =$ Min$\{N, M\}$. When $\Lambda = 10$, $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, and $W(\omega) = 1$, the curve of the maximum phase error versus K_Λ is shown in Fig. 4, and $\Gamma_\Lambda = 2.6$. It is seen in Fig. 4 that when $K_\Lambda > 0$, the phase error increases rapidly with an increasing K_Λ, and when $K_\Lambda < 0$, the phase error is much smaller and has a peak nearby $K_\Lambda = -0.6$. When $K_\Lambda = 0$ or -2, the allpass filter degenerates into a delay $z^{-\Lambda}$ or $z^{-(\Lambda-2)}$. Hence, the interconnection of a delay section and an allpass filter is included in the parallel structure of two allpass filters as a special case.

Example 2: We consider design of a lowpass and highpass filter pair with $N = 11$ and $M = 10$, $\omega_p = 0.4\pi$, and $\omega_s = 0.6\pi$. The weighting function is set to $W(\omega) = 1$ in the passband and stopband. The range of the group delay in which the stable filter can be designed is $9 \le \tau \le 12.6$. We have selected $\tau = 10.5$ and designed $A_N(z)$ and $A_M(z)$. The resulting phase responses and errors are shown in Figs. 5 and 6, respectively. The magnitude and phase responses of $H(z)$ and $G(z)$ are shown in Fig. 7. It is seen that the equiripple magnitude and phase responses are simultaneously obtained. In the above design examples, we needed about 4–6 iterations to obtain the equiripple phase responses.

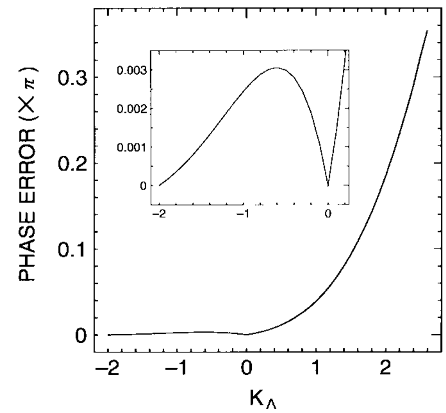

Fig. 4. Maximum phase error versus K_Λ when $\Lambda = 10, \omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, and $W(\omega) = 1$.

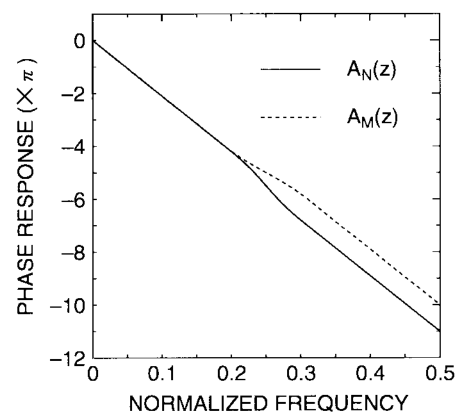

Fig. 5. Phase responses of $A_N(z)$ and $A_M(z)$.

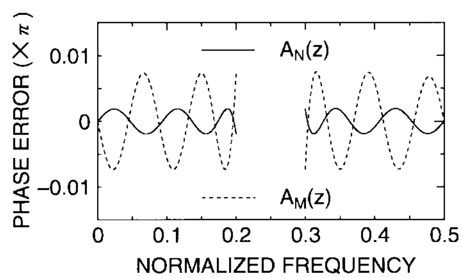

Fig. 6. Phase errors of $A_N(z)$ and $A_M(z)$.

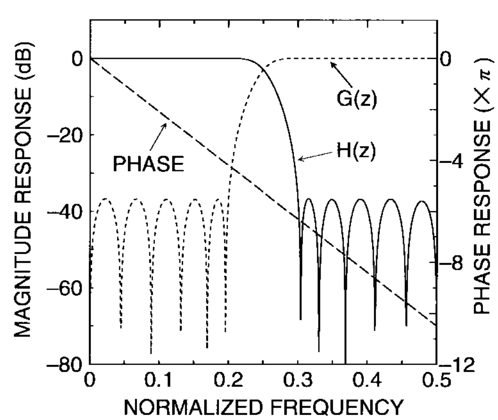

Fig. 7. Magnitude and phase responses of $H(z)$ and $G(z)$.

VI. Conclusions

In this correspondence, we have proposed a new method for designing IIR allpass filters with an equiripple phase response based on the eigenvalue problem. We have proven that the equiripple phase solution is optimal in the Chebyshev sense. By using the Remez exchange algorithm, we have formulated the phase approximation problem of IIR allpass filters in the form of an eigenvalue problem and introduced a new and very simple selection rule for the eigenvalue to be searched for, where the rational interpolation is performed if and only if the real maximum eigenvalue is chosen. Therefore, the solution of the rational interpolation problem can be obtained by finding the eigenvector corresponding to the real maximum eigenvalue, and the optimal filter coefficients are easily computed through a few iterations without any initial guess of the solution. The new design algorithm not only retains the speed inherent in the Remez exchange algorithm but also simplifies the interpolation step because it has been reduced to the computation of the real maximum eigenvalue.

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Design of Efficient FIR Filters for the Amplitude Response: $|1/\omega|$ by Using Universal Weights[1]

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***Abstract*—An efficient design of a linear-phase, FIR structure yielding optimal amplitude response approximating $|1/\omega|$ for midband frequency range has been proposed. Mathematical formulas for computation of weights have been derived. These weights turn out to be universal. Using this property, a versatile structure performing optimally for various orders has been proposed.**

***Index Terms*—FIR filters, integrators, optimum filters.**

I. Introduction

In a number of signal processing systems, we are required to approximate the ideal amplitude response: $|1/\omega|$ over the frequency range $0 < \omega \le \pi$. Such an ideal amplitude response may be called a "zerophase integrator" or even an "integrating Hilbert transformer." [Note that an ideal integrator has the frequency response $1/(j\omega)$]. Typically, in instrumentation, where the transducer senses the acceleration/velocity, the displacement is computed by double/single integration [1]. In biomedical measurements, the essential requirement of integration is well known. There are host of other applications where the realization or approximation of the amplitude response $|1/\omega|$ is called for.

Digital filters are obviously preferred to the analog ones. The finite impulse response (FIR) filters are extensively used due to their guaranteed stability and linear phase characteristics [2]. Theoretically, the ideal amplitude response $|1/\omega|$ cannot be realized by any FIR filter due to a pole at $\omega = 0$. However, if we confine ourselves to

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[1]The authors have filed a patent application based on the design presented in this paper.