Design of FIR Nyquist Filters with Low Group Delay

Xi Zhang and Toshinori Yoshikawa

***Abstract*— A new method is proposed for designing FIR Nyquist filters with zero-crossing impulse response and low group delay. It is first shown that FIR Nyquist filters that satisfy the zero-crossing time-domain condition have a frequency response property where both the magnitude and phase responses in the passband are dependent on the stopband response. Therefore, the design problem will become a magnitude approximation in the stopband. The proposed procedure is based on the formulation of a linear problem by using the multiple Remez exchange algorithm in the stopband directly. Hence, the filter coefficients can be computed by solving linear equations, and the optimal solution with an equiripple stopband response is easily obtained after applying an iteration procedure. Although the proposed Nyquist filters have an approximate linear phase response, its group delay is lower than the conventional FIR Nyquist filters. The proposed procedure is computationally efficient because it only solves a set of linear equations. Finally, the characteristics of the low-delay FIR Nyquist filters are examined, and the performance is compared with the conventional FIR Nyquist filters.**

***Index Terms*—FIR Nyquist filter, low group delay, Remez exchange algorithm.**

I. Introduction

In recent years, Nyquist filters have been found numerous applications in perfect reconstruction filter banks, nonuniform sampling, interpolation filters, and so on. Its impulse response is required to be exactly zero-crossing at the Nyquist rate, except for one point. Nyquist filters can be realized by using either FIR or IIR filters. FIR Nyquist filters can be designed with an exact linear phase, and the design problem has been exhaustively studied in [1]–[14]. However, when the sharp magnitude specifications are required, higher order FIR filters are generally needed, and a larger delay results. This is because the group delay is equal to half the filter order for the exact linear-phase FIR filters. In some applications of real-time signal processing, a lower delay is generally required. In this paper, we will consider the design of FIR Nyquist filters with low group delay. There are also several design methods for IIR Nyquist filters in [15]–[22]. IIR Nyquist filters have two shortcomings in general; one is the stability that must be considered in the design procedure, and another is that the existing design methods are generally time consuming.

In this correspondence, we propose a new method for designing FIR Nyquist filters with zero-crossing impulse response and low group delay. First, we investigate the frequency response property of FIR Nyquist filters with zero-crossing impulse response. Both the magnitude and phase responses in the passband are dependent on the stopband response, and then, only the stopband response is needed in the approximation. Hence, the design problem will become the minimization of the magnitude error in the stopband. By applying the multiple Remez exchange algorithm in the stopband directly, we formulate the design problem in the form of a linear problem. Therefore, a set of filter coefficients can be computed by solving the linear equations, and the optimal solution with an equiripple stopband response is easily obtained through a few iterations. Although the proposed Nyquist filters have an approximate linear phase response, their group delay is lower than the conventional FIR Nyquist filters. The proposed method is computationally efficient because it only solves a set of linear equations. Finally, we examine the characteristics of the low-delay FIR Nyquist filters and compare the performance with the conventional FIR Nyquist filters to demonstrate the effectiveness of the proposed method.

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The authors are with the Department of Electrical Engineering, Nagaoka University of Technology, Niigata, Japan.

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II. Property of FIR Nyquist Filters

Let the transfer function $H(z)$ of a FIR digital filter of order N be

$$H(z) = \sum_{n=0}^{N} h_n z^{-n} \tag{1}$$

where h_n are real filter coefficients. When $H(z)$ is designed as a Nyquist filter, its impulse response is required to be exactly zero-crossing at the Nyquist rate except for one point K, i.e.,

$$\begin{aligned} h_K &= \frac{1}{M} \\ h_{K+kM} &= 0 \qquad (k = \pm 1, \pm 2, \cdots) \end{aligned} \tag{2}$$

where K and M are integers, and M is the Nyquist interval; K corresponds to the desired group delay in the passband. In the case of FIR filters with exact linear phase, the filter coefficients have to be symmetric, i.e., $h_n = h_{N-n}$, and then, its group delay equals $K = N/2$. Hence, K increases with an increasing filter order N. It results in a larger delay when higher order FIR filters are needed. In some applications of real-time signal processing, a lower delay is generally required. In this correspondence, we consider the design of FIR Nyquist filters with low group delay and without the exact linear-phase constraint. Hence, the proposed Nyquist filters have an approximate linear-phase response. In general, Nyquist filters are required to be lowpass, and the desired magnitude response is given by

$$|H_d(e^{j\omega})| = \begin{cases} 1, & (0 \le \omega \le \omega_p) \\ 0, & (\omega_s \le \omega \le \pi) \end{cases} \tag{3}$$

where the passband and stopband cut-off frequencies are $\omega_p = (1 - \rho/M)\pi$ and $\omega_s = (1 + \rho/M)\pi$, and ρ is a rolloff rate. Let a noncausal shifted version of $H(z)$ be $\hat{H}(z) = z^K H(z)$. By substituting the time-domain condition of (2) into (1), we have

$$\hat{H}(z) = z^K H(z) = \frac{1}{M} + \sum_{\substack{n=0 \\ \neq K+kM}}^{N} h_n z^{K-n} \tag{4}$$

where $\hat{H}(z)$ is more suitable for approximation, whereas $H(z)$ is actually implemented. Therefore, the design problem becomes the approximation of $\hat{H}(z)$ in (4) to the desired magnitude response of (3).

Before designing $\hat{H}(z)$, we investigate the property of FIR Nyquist filters with zero-crossing impulse response. It can be seen from (4) that the frequency response of $\hat{H}(z)$ always satisfies

$$\sum_{m=0}^{M-1} \hat{H}(e^{j(\omega + (2m\pi/M))}) \equiv 1 \tag{5}$$

which means that the sum of the responses at the frequency points $\omega_m = \omega + (2m\pi/M)$ for $m = 0, 1, \cdots, M-1$ are equal to one, regardless of what the filter coefficients h_n are. Due to $\hat{H}(e^{j(2\pi-\omega)}) = \hat{H}^*(e^{j\omega})$, we have

$$\hat{H}(e^{j\omega_0}) = 1 - \sum_{i=1}^{L-1} \hat{H}(e^{j\omega_i}) - \sum_{i=L}^{M-1} \hat{H}^*(e^{j\omega_i}) \quad (0 \le \omega_0 \le \omega_p) \tag{6}$$

where $L = \lfloor M+1/2 \rfloor, \omega_i = (2i\pi/M) + \omega_0$ for $0 \le i \le L-1$, and $\omega_i = (2(M-i)\pi/M) - \omega_0$ for $L \le i \le M-1$. x^* and $\lfloor x \rfloor$ denote the complex conjugate and integer part of x, respectively. This means that the response at ω_0 is dependent on the responses at ω_i $(i = 1, 2, \cdots, M-1)$. If its stopband response is 0, then the frequency response of $\hat{H}(z)$ will be 1 in the passband, i.e., the magnitude response of $H(z)$ is 1, and the phase response is linear, that is, $-K\omega$ in the passband. Therefore, both the passband magnitude and phase error are decided by the stopband error. Let δ_s be the maximum magnitude error in the stopband, and the maximum magnitude and phase errors in the passband are

$$\begin{aligned} \delta_p &\le (M-1)\delta_s \\ \Delta\theta &\le \sin^{-1}(M-1)\delta_s. \end{aligned} \tag{7}$$

In practical designs, δ_p and $\Delta\theta$ are usually much smaller than these upper limits. Since δ_p and $\Delta\theta$ are guaranteed to be relatively small for a small value of δ_s, the filter design can concentrate on shaping the stopband response. It can also be explained according to the zero locations. Since there are $I\ (= N - \lfloor K/M \rfloor - \lfloor N-K/M \rfloor)$ unknown coefficients h_n in (4), $H(z)$ has I independent zeros that are used to provide the desired stopband response. These independent zeros must exist on the unit circle to minimize the stopband error, whereas the remaining zeros off the unit circle are used for satisfying the time-domain condition of (2) so that the passband response is naturally formed. In the following, we will directly apply the multiple Remez exchange algorithm in the stopband to design FIR Nyquist filters with low group delay.

III. Design of FIR Nyquist Filters

In this section, we describe the complex Chebyshev approximation of FIR Nyquist filters with zero-crossing impulse response and low group delay by directly using the multiple Remez exchange algorithm in the stopband.

A. Initial Choice

It is known that $H(z)$ has I independent zeros that have to be located on the unit circle to minimize the stopband magnitude error. First, we assume that an initial location of I independent zeros on the unit circle is

$$z_i = e^{\pm j\overline{\omega}_i} \qquad (\omega_s, < \overline{\omega}_1 < \overline{\omega}_2 < \cdots < \overline{\omega}_{J_1} \le \pi) \tag{8}$$

where $J_1 = \lfloor I + 1/2 \rfloor, \overline{\omega}_{J_1} \ne \pi$ when I is even, and $\overline{\omega}_{J_1} = \pi$ when I is odd. This stems from the fact that the zeros on the unit circle must occur in complex conjugate pairs, except for $z = \pm 1$. A possible choice of $\overline{\omega}_i$ is to pick these frequencies to be equally spaced in the stopband. Other distributions may also be preferred. From (4), we have

$$\hat{H}(e^{\pm j\overline{\omega}_i}) = \frac{1}{M} \sum_{\substack{n=0 \\ \ne K+kM}}^{N} h_n e^{\pm j(K-n)\overline{\omega}_i} = 0 \tag{9}$$

which can be divided into the real and imaginary parts as

$$\sum_{\substack{n=0 \\ \ne K+kM}}^{N} h_n \cos(K-n)\overline{\omega}_i = -\frac{1}{M} \quad (i = 1, 2, \cdots, J_1) \tag{10}$$

and

$$\sum_{\substack{n=0 \\ \ne K+kM}}^{N} h_n \sin(K-n)\overline{\omega}_i = 0 \qquad (i = 1, 2, \cdots, J_2) \tag{11}$$

where $J_2 = \lfloor I/2 \rfloor$ because when I is odd, (9) has only the real part and not the imaginary part at $\overline{\omega}_{J_1} = \pi$. Therefore, there are a total of I equations in (10) and (11), whether I is odd or even. By solving the above linear equations, we can easily obtain a set of initial filter coefficients h_n whose independent zeros are located on the unit circle. Since the initial locations of the independent zeros are manually selected, the obtained magnitude response may not be equiripple in the stopband. Therefore, we directly apply the multiple Remez exchange algorithm in the stopband to obtain an equiripple stopband response.

B. Formulation

Since all the initial independent zeros are guaranteed to be located on the unit circle, we can search the magnitude response in the stopband, and get $J_2 + 1$ extremal frequencies ω_i as

$$\omega_s = \omega_0 < \omega_1 < \cdots < \omega_{J_2} \le \pi \tag{12}$$

where $\omega_{J_2} < \pi$ when I is odd due to $\overline{\omega}_{J_1} = \pi$, and $\omega_{J_2} = \pi$ when I is even. We apply the multiple Remez exchange algorithm in the stopband and formulate the condition for $\hat{H}(e^{j\omega})$ as

$$\hat{H}(e^{j\omega_i}) = \delta_s e^{j\theta(\omega_i)} \tag{13}$$

where $\delta_s(>0)$ is the stopband magnitude error to be minimized, and $\theta(\omega_i)$ is the phase response of $\hat{H}(z)$ at ω_i and computed by using the previous filter coefficients h_n. Substituting $\hat{H}(e^{j\omega})$ of (4) into (13), we divide (13) into the real and imaginary parts as

$$\sum_{\substack{n=0 \\ \ne K+kM}}^{N} h_n \cos(K-n)\omega_i - \delta_s \cos\theta(\omega_i) = -\frac{1}{M} \qquad (i = 0, 1, \cdots, J_2) \tag{14}$$

and

$$\sum_{\substack{n=0 \\ \ne K+kM}}^{N} h_n \sin(K-n)\omega_i - \delta_s \sin\theta(\omega_i) = 0 \qquad (i = 0, 1, \cdots, J_1 - 1). \tag{15}$$

Note that (13) has no imaginary part at $\omega_{J_2} = \pi$ when I is even. It is clear that there are a total of $I + 1$ equations in (14) and (15) whether I is odd or even, and hence, we can obtain a set of new filter coefficients by solving the above linear equations. By using the obtained filter coefficients, we compute the frequency response of $\hat{H}(z)$ in the stopband to find the peak frequency points Ω_i and compute the corresponding phase response $\theta(\Omega_i)$. As a result, the obtained peak frequency points Ω_i may not be consistent with the extremal frequencies ω_i. We then use the obtained peak frequency points as the extremal frequencies in the next iteration and solve the linear equations of (14) and (15) to obtain h_n again. The algorithm is iterated until the equiripple stopband response is attained. The design algorithm is shown in detail as follows.

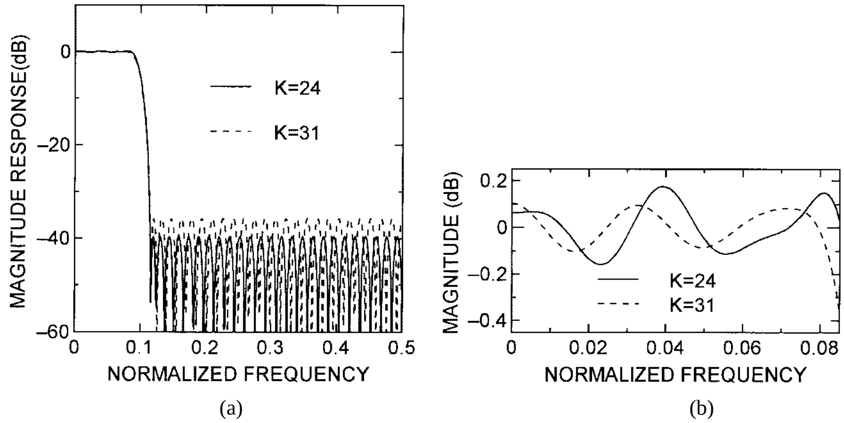

Fig. 1. Magnitude responses of FIR Nyquist filters: (a) Log magnitude in decibels. (b) Passband detail.

C. Design Algorithm

Procedure { Design algorithm of FIR Nyquist filters }
Begin

1) Read the design specifications N, M, K and ρ.
2) Select the initial locations $\overline{\omega}_i$ (for $i = 1, 2, \cdots, J_1$) of the independent zeros equally spaced in the stopband as shown in (8).
3) Solve the linear equations of (10) and (11) to obtain a set of initial filter coefficients h_n.
4) Compute the frequency response of $\hat{H}(z)$ by using the initial coefficients, then search for the peak frequencies Ω_i for $i = 0, 1, \cdots, J_2)$ in the stopband and compute the corresponding phase $\theta(\Omega_i)$.

Repeat

5) Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, J_2$.
6) Solve the linear equations of (14) and (15) to obtain a set of filter coefficients h_n.
7) Compute the frequency response of $\hat{H}(z)$, then search for the peak frequencies Ω_i for $i = 0, 1, \cdots, J_2)$ in the stopband and compute the corresponding phase $\theta(\Omega_i)$.

Until Satisfy the following condition for a prescribed small constant ϵ:
$\{|\Omega_i - \omega_i| \le \epsilon \qquad (\text{for } i = 0, 1, \cdots, J_2)\}$

End.

IV. Design Example

In this section, we present one design example to demonstrate the effectiveness of the proposed method and then examine the characteristics of the low-delay FIR Nyquist filters and compare the performance with the exact linear-phase FIR Nyquist filter.

Example: We consider the design of FIR Nyquist filter with the following specifications: $N = 62, M = 5, K = 24$, and $\rho = 0.15$. The magnitude response and group delay obtained by using the proposed method are shown in the solid line in Figs. 1 and 2, respectively. It is clear in Fig. 1 that the stopband magnitude response is equiripple. Since the time-domain condition of (2) has been included in (4), the resulting impulse response must be exactly zero crossing. For comparison purposes, we have also designed a FIR Nyquist filter with $K = N/2 = 31$, which is an exact linear-phase filter. The magnitude response and group delay of $K = 31$ are shown in the dashed line in Figs. 1 and 2, respectively. It is clear in Fig. 2 that although the filter of $K = 24$ has an approximate linear-phase response, its group delay is lower than that of $K = 31$. The stopband attenuations of two filters with $K = 24$ and $K = 31$ are 39.6 and 35.7 dB, respectively, and there is difference of 3.9 dB, whereas their maximum magnitude errors are 0.0202 and 0.0429 in the passband. We have designed many filters with various K and investigated the influence of K on the frequency response. The chart of the maximum magnitude and phase errors versus K is shown in Fig. 3. It is seen in Fig. 3 that the magnitude and phase error curves are symmetric to $K = N/2 = 31$, i.e., two filters with K and $N - K$ have the same magnitude and phase errors. This is because the zeros of two filters satisfy the mirror-image relation with respect to the each other's unit circle. The maximum magnitude error in the stopband periodically varies with a period M and increases as K deviates from $K = N/2 = 31$. When $K = N/2 \pm kM$, the stopband magnitude error attains to the peak; then, a smaller stopband error can be acquired nearby $K = N/2 \pm kM \pm M/2$. The maximum magnitude and phase errors in the passband also increase as K deviates from $K = N/2 = 31$. In Table I, we have summarized the zero distribution of FIR Nyquist filters with various K. It can be seen that the exact linear-phase filter with $K = N/2 = 31$ has six zeros both inside and outside the unit circle and 50 zeros on the unit circle. As K decreases to $K = 30$, one zero outside the unit circle moves to $z = \infty$ due to $h_0 = 0$; then, there are only five zeros outside the unit circle. When $K = 29$ and $K = 28$, these filters have one more zero on the unit circle than other filters. This is why the filters with $K = N/2 \pm kM \pm M/2$ have a smaller stopband error. As K decreases to $K = 27$, one zero on the unit circle moves to the origin $z = 0$ and cancels the other out with the poles at the origin due

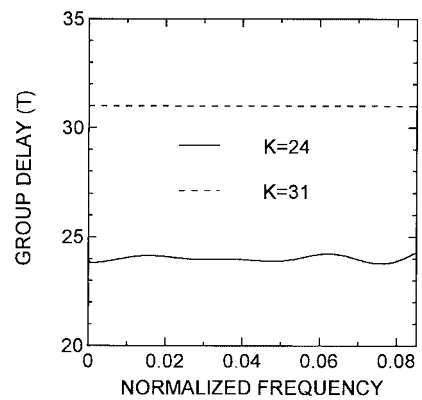

Fig. 2. Group delay responses of FIR Nyquist filters.

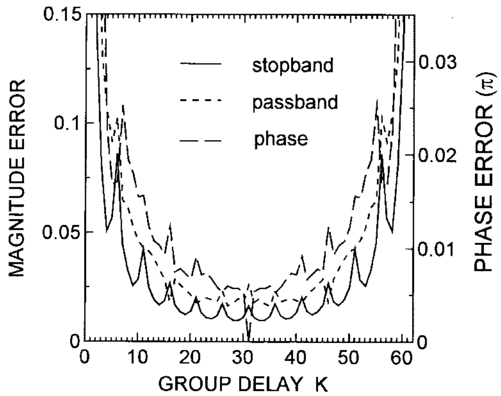

Fig. 3. Plot of maximum magnitude and phase errors versus group delay K.

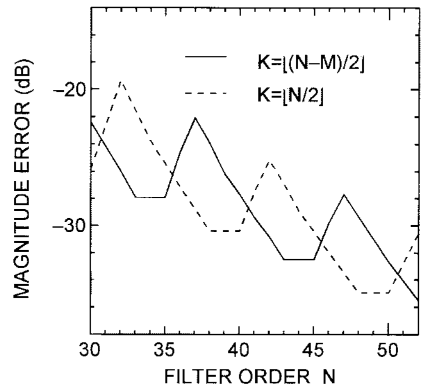

Fig. 4. Plot of maximum magnitude error in stopband versus filter order N with $M = 5$ and $\rho = 0.15$.

TABLE I
ZERO DISTRIBUTION OF FIR NYQUIST FILTERS OF ORDER $N = 62$ WITH $M = 5$, AND $\rho = 0.15$

Group delay K	Number of zeros inside the unit circle	Number of zeros on the unit circle	Number of zeros outside the unit circle
$K = 31$	6	50	6
$K = 30$	6	50	5 $(h_0 = 0)$
$K = 29$	6	51	5
$K = 28$	6	51	5
$K = 27$	6 $(h_N = 0)$	50	5
$K = 26$	7	50	5

to $h_N = 0$; then, the filter degenerates into one of order $N-1$. When $K = 26$, this filter has two more zeros inside the unit circle than it does on the outside. Therefore, as K decreases, each of the zeros outside the unit circle moves first to $z = \infty$, then on the unit circle, then the origin $z = 0$, and, finally, into the inside of the unit circle. When K increases, the zeros just inversely move from the inside to the outside. The movement of the zeros decides the variability on the frequency response (e.g., group delay). Next, we consider the influence of N on the maximum magnitude and phase errors. It is seen in Fig. 3 that the maximum magnitude error in the stopband is larger at $K = N/2$. Here, we ask whether the stopband error always attains the peak at $K = N/2$. To answer this question, we have designed many filters with various N and found that there are a total of $2M$ types of error patterns, i.e., $N \bmod \{2M\} = 0, 1, \cdots, 2M-1$, where $N \bmod \{M\} = N - M\lfloor N/M \rfloor$. We found in all the error patterns that the stopband maximum error periodically varies with a period M and increases as K deviates from $K = N/2$. The varying amplitude becomes minimum when $0 \leq N \bmod \{M\} \leq 1$ and attains a maximum when $\lfloor M/2 \rfloor \leq N \bmod \{M\} \leq \lfloor M/2 \rfloor + 1$. When $1 \leq N \bmod \{2M\} \leq M$, the stopband error attains the peak at $K = N/2$, as shown in Fig. 3, whereas it becomes minimum at $K = N/2$, when $M+1 \leq N \bmod \{2M\} \leq 2M$. Therefore, we can conclude in Fig. 4 that Nyquist filters have a smaller stopband error near $K = N/2 \pm kM \pm M/2$ when $1 \leq N \bmod \{2M\} \leq M$ and near $K = N/2 \pm kM$ when $M+1 \leq N \bmod \{2M\} \leq 2M$. Since the passband frequency response in the proposed method is dependent on the stopband response that is minimized, the passband magnitude and phase errors are much more complicated than the stopband error and will need further investigation.

V. CONCLUSION

In this correspondence, we have proposed a new design method of FIR Nyquist filters with zero-crossing impulse response and low group delay. We have shown that when FIR Nyquist filters satisfy the zero-crossing time-domain condition, the sum of the frequency responses at some related frequencies are equal to one, regardless of what the filter coefficients are. Therefore, both the magnitude and phase responses in the passband are dependent on the stopband response, and then, the design problem has become the minimization of the stopband magnitude error. By applying the multiple Remez exchange algorithm in the stopband directly, we have formulated the design problem in the form of linear problem. Hence, the filter coefficients can be computed by solving a set of linear equations, and the optimal solution with an equiripple stopband response is easily obtained through a few iterations. Although the proposed Nyquist filters have an approximate linear-phase response, its group delay is lower than the exact linear-phase FIR Nyquist filters. The proposed method is computationally efficient because it needs only the solution of a set of linear equations.

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Repeated Filtering in Consecutive Fractional Fourier Domains and Its Application to Signal Restoration

M. Fatih Erden, M. Alper Kutay, and Haldun M. Ozaktas

***Abstract*—Filtering in a single time domain or in a single frequency domain has recently been generalized to filtering in a single fractional Fourier domain. In this correspondence, we further generalize this to repeated filtering in consecutive fractional Fourier domains and discuss its applications to signal restoration through an illustrative example.**

I. Introduction

Fig. 1(a) shows multiplication of an input signal $f_{\rm in}(u)$ with a multiplicative filter $h(u)$ to obtain the output signal $f_{\rm out}(u)$. We call this operation *multiplicative filtering in the time domain*. Similarly, we refer to the operation in Fig. 1(b) as *multiplicative filtering in the frequency domain* (note that we are using the same dummy variable u in both the time and frequency domains). In Fig. 1(c), we show multiplicative filtering in the ath-order fractional Fourier transform domain. In this configuration, first the ath fractional Fourier transform of the input is obtained, and then, a multiplicative filter $h(u)$ is applied in this domain. Finally, the resulting waveform is transformed with order $-a$ in order to obtain the output profile in the time domain (the $-a$th transform is the inverse of the $+a$th transform). The ath-order fractional Fourier transformation reduces to the identity operation for $a = 0$; therefore, Fig. 1(c) reduces to Fig. 1(a) for $a = 0$. Likewise, it reduces to the ordinary Fourier transformation for $a = 1$ so that in this case, Fig. 1(c) reduces to Fig. 1(b). In [1]–[4] it is shown that the added degree of freedom offered by the order parameter a allows improved performance (e.g., smaller mean-square error) in a variety of circumstances including restoration of time-varying signals degraded by nonstationary noise. Furthermore, since both the digital [5] and optical [6]–[8] implementations of the fractional Fourier transformation do not imply extra work compared with the ordinary Fourier transformation, these improvements are achieved at no additional cost.

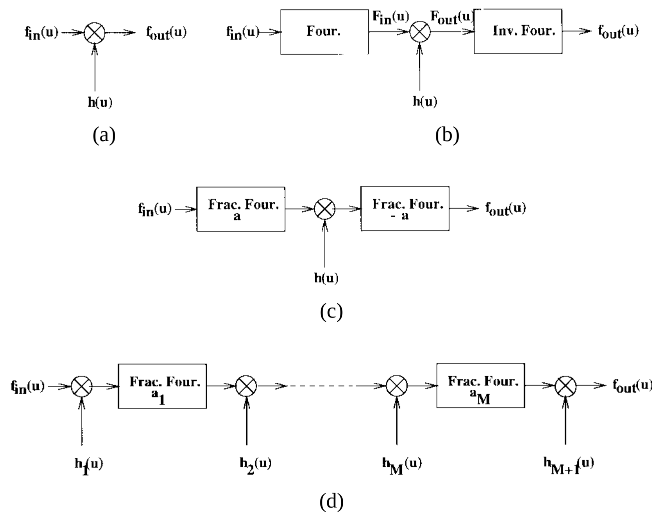

Fig. 1. Configurations that correspond to (a) filtering in the time domain, (b) filtering in the frequency domain, (c) filtering in a single fractional Fourier domain, (d) repeated filtering in consecutive fractional Fourier domains.

We can further generalize the concept of single fractional Fourier domain filtering [Fig. 1(c)] to repeated filtering in consecutive fractional Fourier domains [Fig. 1(d)]. Here, we apply the first filter in the 0th fractional domain (the time domain), the second filter in the a_1st fractional domain, the third filter in the $(a_1 + a_2)$nd fractional domain, and so on. This generalization was first mentioned in [1] and [8]. However, in those papers, the problem of finding the filter profiles for a specified application has not been addressed. In this correspondence, we discuss the applications of this filtering configuration to signal restoration. More specifically, we seek the optimal filter profiles resulting in the minimum mean-square estimate of the original signal.

II. Fractional Fourier Transformation

The ath-order fractional Fourier transform $p_a(u)$ of $p(u)$ is defined for $0 < |a| < 2$ as

$$p_a(u) = \sqrt{1 - j\cot\phi}\int_{-\infty}^{\infty} \exp[j\pi(\cot\phi u^2 - 2\csc\phi u u' + \cot\phi u'^2)]p(u')\,du' \qquad (1)$$

where $\phi = a\pi/2$. The kernel is defined separately for $a = 0$ and $a = \pm 2$ as $B_0(u,u') \equiv \delta(u - u')$ and $B_{\pm 2}(u,u') \equiv \delta(u + u')$, respectively [9]. The definition is easily extended outside the interval $[-2, 2]$ through $\mathcal{F}^{4i+a}\hat{q} = \mathcal{F}^a\hat{q}$ for any integer i.

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M. F. Erden is with Massana Ltd., Dublin, Ireland.
M. A. Kutay is with the Department of Electrical and Computer Engineering, Drexel University, Philadelphia, PA 19104 USA.
H. M. Ozaktas is with the Department of Electrical Engineering, Bilkent University, Ankara, Turkey.
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