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# COMPLEX CHEBYSHEV APPROXIMATION FOR IIR DIGITAL FILTERS

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## ABSTRACT

This paper presents an efficient method for designing complex IIR digital filters in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm. Hence, the filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and then the complex Chebyshev approximation is attained through a few iterations starting with a given initial guess. The proposed algorithm is computationally efficient. One example is designed and compared with one proposed by Chen and Parks. It is shown that the results obtained by using the method proposed in this paper are better than the conventional methods.

## 1. INTRODUCTION

It is well-known [1] that the Remez exchange algorithm is an efficient tool for designing FIR filters with linear phase, where the design problem is a real Chebyshev approximation. In many applications such as equalization and beamforming, the design of filters with arbitrary magnitude and phase responses is needed, which results in a complex Chebyshev approximation problem [1]–[6]. The Remez exchange algorithm has also been generalized to the complex case and used to design complex FIR filters [2],[3]. Compared with FIR digital filters, IIR filters tend to be of much lower order for meeting the same specifications. However, IIR filter design is more difficult than FIR design because it is a rational approximation. In [7] and [8], the Remez exchange algorithm has been applied to the real Chebyshev approximation for IIR filters, where the interpolation problem has been reduced to the generalized eigenvalue problem, thus the solution can be easily obtained by finding the absolute minimum eigenvalue in most cases. In this paper, we wish to generalize the method in [7] to the complex Chebyshev approximation of IIR filters. Several methods have been suggested to design complex IIR filters in the complex domain also [4]–[6]. However, the major disadvantage of these methods is quite computationally expensive.

In this paper, we consider the complex Chebyshev approximation problem of IIR digital filters, and propose an efficient method to attain the specified magnitude and phase responses in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm. Hence, the filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and then the complex Chebyshev approximation is attained through a few iterations starting with a given initial guess. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm, but also simplifies the interpolation step. Finally, one example is presented and compared with the method proposed by Chen and parks in [5]. It is shown that the results obtained by using the proposed method are better than the conventional methods.

## 2. PROBLEM STATEMENT

Let $H(z)$ be the transfer function of an IIR digital filter with numerator degree $N$ and denominator degree $M$:

$$H(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}}, \tag{1}$$

where filter coefficients $a_n, b_m$ are complex, and $b_0 = 1$. The frequency response of $H(z)$ is generally a complex-valued function of the normalized frequency $\omega$:

$$H(e^{j\omega}) = \frac{\sum_{n=0}^{N} a_n e^{-jn\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}} = \frac{N(\omega)}{D(\omega)}. \tag{2}$$

The complex Chebyshev approximation problem may be briefly stated as follows. Let $H_d(e^{j\omega})$ be the desired frequency response:

$$H_d(e^{j\omega}) = |H_d(e^{j\omega})| e^{j\theta_d(\omega)} \qquad (\omega \in R), \tag{3}$$

where $|H_d(e^{j\omega})|$ is the desired magnitude response, $\theta_d(\omega)$ the desired phase response and $R \subset (-\pi, \pi]$ the interest bands (e.g., passband and stopband). The approximation problem consists in finding the filter coefficients $a_n, b_m$ that will minimize the Chebyshev norm

$$||E(\omega)|| = \max_{\omega \in R} |E(\omega)| \tag{4}$$

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of the weighted error

$$E(\omega) = W(\omega)[H(e^{j\omega}) - H_d(e^{j\omega})] \tag{5}$$

among all possible choices of $a_n, b_m$. The weighting $W(\omega)$ must be a real and strictly positive function.

In order to guarantee the filter causality and stability, the poles are required to locate inside the unit circle. It is known in [5] that the optimal complex Chebyshev approximation may not exist when the poles are restricted inside the unit circle. In some applications such as image processing, it is not necessary for the filter to be causal since the signal length is finite. Therefore, the constraint can be relaxed and only the stability remains to be considered. In this case, the poles will be required only not to locate on the unit circle. It was pointed out in [5] that there is no guarantee of the uniqueness of the complex Chebyshev approximation problem, and the number of the optimal approximation may be arbitrarily large. The characterization of the optimal rational complex Chebyshev approximation is available as sufficient conditions for the general approximation without pole restrictions. One sufficient condition is that $|E(\omega)|$ has at least $N+M+2$ extremal points [5]. In the following, we will make use of this sufficient condition in the problem formulation without any pole restrictions. The filter stability issue is addressed in Section **6**.

## 3. FORMULATION BASED ON EIGENVALUE PROBLEM

In this section, we describe the design of IIR digital filters based on the eigenvalue problem by using the Remez multiple exchange algorithm. Our aim is to find the filter coefficients $a_n, b_m$ in such a way that the error function in Eq.(5) satisfies

$$|E(\omega)| \leq \delta_{max} \qquad (\omega \in R), \tag{6}$$

where $\delta_{max}$ ($> 0$) is a maximum error to be minimized.

To solve the above complex Chebyshev approximation problem, we use the Remez multiple exchange algorithm and formulate the condition for $H(e^{j\omega})$ of Eq.(2) in the form of a generalized eigenvalue problem. By selecting $N+M+2$ extremal frequencies $\omega_i$ ($i = 0, 1, \cdots, N+M+1$) in the bands $R$, we formulate $H(e^{j\omega})$ as

$$E(\omega_i) = W(\omega_i)[H(e^{j\omega_i}) - H_d(e^{j\omega_i})] = \delta e^{j\theta_e(\omega_i)}, \tag{7}$$

where $\delta$ is error magnitude and $\theta_e(\omega)$ is phase of the error function $E(\omega)$. Note that the denominator polynomial $D(\omega)$ of $H(e^{j\omega})$ must satisfy

$$D(\omega) = \sum_{m=0}^{M} b_m e^{-jm\omega} \neq 0 \qquad (\omega \in R). \tag{8}$$

Substituting Eq.(2) into Eq.(7), we get

$$N(\omega_i) - H_d(e^{j\omega_i})D(\omega_i) = \delta \frac{e^{j\theta_e(\omega_i)}}{W(\omega_i)} D(\omega_i). \tag{9}$$

Then we rewrite Eq.(9) in the matrix form as

$$\boldsymbol{P}\boldsymbol{a} = \delta\, \boldsymbol{Q}\boldsymbol{a}, \tag{10}$$

where $\boldsymbol{a} = [a_0, a_1, \cdots, a_N, b_0, b_1, \cdots, b_M]^T$, and the elements of the matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ are given by

$$P_{mn} = \begin{cases} e^{-jn\omega_m} & (n = 0, 1, \cdots, N) \\ -H_d(e^{j\omega_m})e^{-j(n-N-1)\omega_m} & \\ & (n = N+1, \cdots, N+M+1) \end{cases} \tag{11}$$

$$Q_{mn} = \begin{cases} 0 & (n = 0, 1, \cdots, N) \\ \dfrac{e^{j(\theta_e(\omega_m)-(n-N-1)\omega_m)}}{W(\omega_m)} & \\ & (n = N+1, \cdots, N+M+1) \end{cases} \tag{12}$$

Once $N+M+2$ extremal frequencies $\omega_i$ and its phase $\theta_e(\omega_i)$ are given, it is seen from Eqs.(11) and (12) that the elements of the matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ are known. Therefore, it should be noted that Eq.(10) corresponds to a generalized eigenvalue problem, i.e., $\delta$ is an eigenvalue and $\boldsymbol{a}$ is a corresponding eigenvector. In order to minimize $\delta$, we must find the absolute minimum eigenvalue by solving the above eigenvalue problem [7], which can be done efficiently by using the iterative power method, so that the corresponding eigenvector gives a set of filter coefficients. By using the obtained filter coefficients, we compute the error function $E(\omega)$ and search for the peak frequencies $\Omega_i$ of $|E(\omega)|$ in the bands $R$. It is found that the obtained $|E(\omega)|$ may not be equiripple. Thus, we use these peak frequencies $\Omega_i$ as the extremal frequencies $\omega_i$ in next iteration and compute the phase response of $E(\omega)$ to obtain $\theta_e(\omega_i)$. The eigenvalue problem is then solved to obtain a set of filter coefficients again. The above procedure is iterated until the equiripple response is attained. Note that in the above-mentioned iteration procedure, an initial value of the extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$ will be needed. The selection of the initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$ will directly influence the convergence of the iteration procedure. In the following, we will discuss how to select the initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$.

## 4. SELECTION OF INITIAL VALUE

In the proposed iteration procedure, arbitrarily selecting a set of initial extremal frequencies $\omega_i$ and its phase $\theta(\omega_i)$ cannot guarantee the algorithm to converge to the optimal solution. Hence, it is very important how to select the initial value. We must select an initial value enough close to the optimal solution to guarantee the convergence of the algorithm. It is known that the aim of the filter design is to minimize the error $\delta$. Therefore, we firstly select $N+M+1$ frequency points $\bar{\omega}_i$ within $R$ and then force $E(\omega)$ equal to zero at these frequency points $\bar{\omega}_i$:

$$E(\bar{\omega}_i) = W(\bar{\omega}_i)[H(e^{j\bar{\omega}_i}) - H_d(e^{j\bar{\omega}_i})] = 0, \tag{13}$$

where a possible choice of $\bar{\omega}_i$ is to pick these frequency points equally spaced in the bands $R$. Other distributions may also be preferred to decrease the number of iterations. The denominator polynomial $D(\omega)$ must satisfy Eq.(8) also.

Substituting Eq.(2) into Eq.(13), we have

$$\sum_{n=0}^{N} a_n e^{-jn\bar{\omega}_i} - H_d(e^{j\bar{\omega}_i}) \sum_{m=0}^{M} b_m e^{-jm\bar{\omega}_i} = 0, \qquad (14)$$

which is a set of linear equations. Since $b_0 = 1$, there is a unique solution. Hence, we can obtain an initial solution of filter coefficients $a_n, b_m$ by solving the linear equations of Eq.(14). By using the obtained initial filter coefficients, we compute $E(\omega)$ and search for the peak frequencies of $|E(\omega)|$ in the bands $R$. Since we have forced $E(\omega)$ equal to zero at $N+M+1$ frequency points, there always exist more than $N+M+1$ peak frequencies. We then select the first $N+M+2$ peak frequencies $\Omega_i$ in descending order of magnitude as the initial extremal frequencies $\omega_i$ and compute the phase response of $E(\omega)$ to obtain $\theta(\omega_i)$. The design algorithm is shown as follows.

## 5. DESIGN ALGORITHM

**Procedure** {Design Algorithm of IIR Digital Filters}
**Begin**

1. Read $N$, $M$, $H_d(e^{j\omega})$ and $W(\omega)$.
2. Select $N+M+1$ frequency points $\bar{\omega}_i$ equally spaced within $R$.
3. Solve Eq.(14) to obtain an initial solution of filter coefficients $a_n, b_m$.
4. Compute $E(\omega)$ by using the initial filter coefficients, then search for the first $N+M+2$ peak frequencies $\Omega_i$ in $R$ and compute its phase $\theta(\Omega_i)$.

**Repeat**

5. Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, N+M+1$.
6. Compute $\boldsymbol{P}$ and $\boldsymbol{Q}$ by using Eqs.(11) and (12), and then find the absolute minimum eigenvalue of Eq.(10) to obtain a set of filter coefficients $a_n, b_m$.
7. Compute $E(\omega)$, then search for the peak frequencies $\Omega_i$ in $R$ and compute its phase $\theta(\Omega_i)$.

**Until** Satisfy the following condition for a prescribed small constant $\epsilon$:

$$\left\{ \sum_{i=0}^{N+M+1} |\Omega_i - \omega_i| \leq \epsilon \right\}$$

**End.**

## 6. STABILITY ISSUE

In the above design algorithm, the obtained filter $H(z)$ may not be stable. The stability must be checked by finding the pole location. To guarantee the filter stability, we have to avoid the poles located on the unit circle, i.e, Eq.(8) must be satisfied for all $\omega$. In section **3**, we have chosen the absolute minimum eigenvalue, which ensures that Eq.(8) is satisfied in the bands $R$. However, Eq.(8) may not be satisfied in the "don't care" band. For example, IIR lowpass filters with nearly linear phase always have a pair of poles in the transition band near the passband edge. This pair of poles maybe move toward the unit circle as the desired group delay varies [5]. It was shown in [5] that the stability of $H(z)$ is mainly dependent on the specifications, i.e., the filter degree $N, M$ and the desired frequency response $H_d(e^{j\omega})$. Therefore, the specifications should be carefully chosen to guarantee the filter stability. See [5] in detail.

## 7. DESIGN EXAMPLE

For comparison purposes, we have designed a real-valued IIR lowpass filter with the same specifications as *Example 1* in [5] by using the proposed method. The specification is $N = M = 4$,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j5\omega} & (0 \leq \omega \leq 0.2\pi) \\ 0 & (0.4\pi \leq \omega \leq \pi) \end{cases}.$$

The weighting is $W(\omega) = 1$ in both passband and stopband. The initial guess of $\bar{\omega}_i$ is shown in Fig.1. Note that since the frequency response of the real filter is complex conjugate between positive and negative frequencies, we only show the positive one. We then selected the initial extremal frequencies $\omega_i$ as shown in Fig.1, and obtained the equiripple solution after six iterations. The resulting magnitude response of $E(\omega)$ is shown in Fig.2, and the maximum error is $\delta_{max} = 0.0234$ whereas $\delta_{max} = 0.0420$ in [5]. The magnitude response and group delay of $H(z)$ are shown in Fig.3 and Fig.4, respectively. The results in [5] are also shown in the dotted line for comparison. The magnitude error is 0.0233 in passband and 0.0234 ($32.6dB$) in stopband, while the error in [5] is 0.0420 and 0.0420 ($27.5dB$) respectively. The group delay in passband is between 4.83 and 5.97, and its maximum deviation from the desired group delay is 0.97 in the passband edge. In [5], the group delay is between 4.65 and 6.34 and its maximum deviation is 1.34. The pole-zero location of the obtained filter is shown in Fig.5 and it is clear that it is causal and stable. To examine the relationship between the specifications and stability, we show the plot of the maximum pole radius versus group delay in Fig.6. It is seen in Fig.6 that when the group delay is 2.5, the maximum pole radius is equal to 1, i.e., this pair of poles locate on the unit circle, thus the filter is unstable. When the group delay is larger than 2.5, the maximum pole radius is smaller than 1, then the filter becomes causal and stable. Therefore, we should specify a larger group delay to guarantee the causality and stability.

## 8. CONCLUSION

In this paper, we have proposed an efficient method for designing complex IIR digital filters in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm. Hence, the filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and the complex Chebyshev approximation is attained through a few iterations starting with a given initial guess. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm, but also simplifies the interpolation step. It is shown through design example that the results obtained by using the method proposed in this paper are better than the conventional methods.

## 9. REFERENCES


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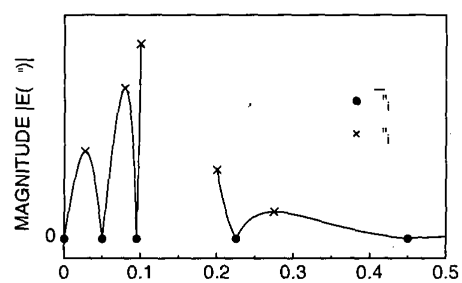


**Fig.1** Initial guess.

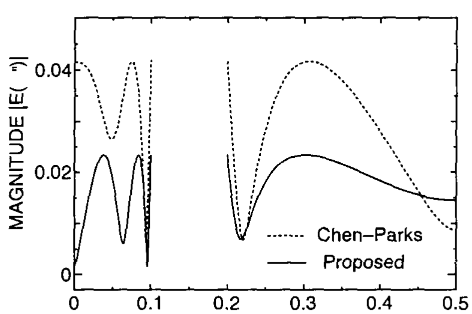


**Fig.2** Magnitude response of $E(\omega)$.

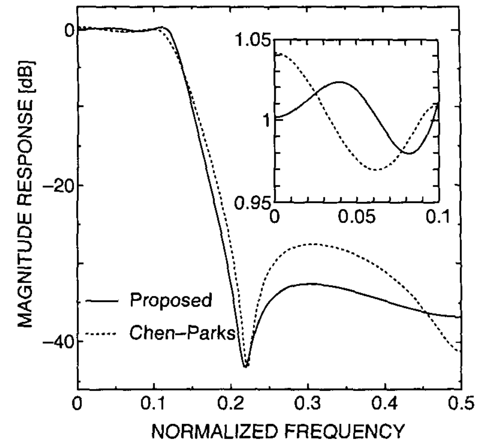


**Fig.3** Magnitude response of $H(z)$.

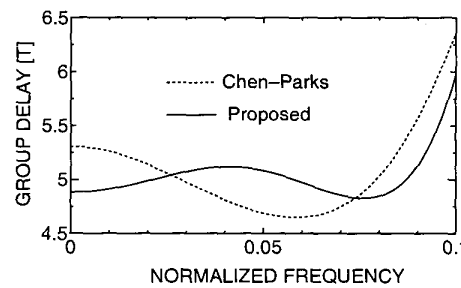


**Fig.4** Group delay of $H(z)$.

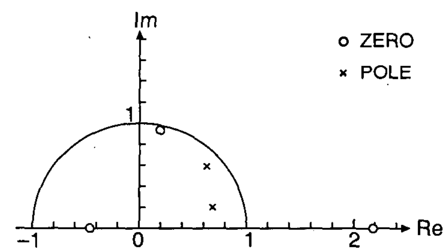


**Fig.5** Pole-zero location of $H(z)$.

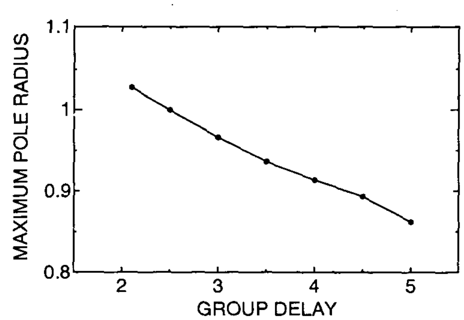


**Fig.6** Maximum pole radius versus group delay.