Design of Q-Shift Filters With Flat Group Delay

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***Abstract*— Q-shift filters have been proposed by Kingsbury for DTCWTs (Dual Tree Complex Wavelet Transforms), and are required to have linear phase responses. This paper proposes a new method for designing Q-shift filters with flat group delay responses. The proposed design method make use of the transfer function proposed by Gopinath, which satisfies both the specified degree of flatness for the group delay and the specified number of vanishing moments, i.e., the specified number of zeros at $z = -1$. Therefore, the design is reduced to how to force the filter to satisfy the condition of orthogonality. The design problem is linearized, and then an iterative procedure is used to obtain the filter coefficients. Finally, some examples are presented to demonstrate the effectiveness of the proposed design method.**

I. Introduction

The Dual Tree Complex Wavelet Transforms (DTCWTs) were originally proposed by Kingsbury in [3], and have been found to be successful in many applications of signal processing and image processing [3]∼[11]. DTCWTs have the following significant properties over DWTs (Discrete Wavelet Transforms); approximate shift invariance, and good directional selectivity for multidimensional signals. It has been shown in [6] that two scaling lowpass filters are required to satisfy the half-sample delay condition, thus the corresponding wavelet bases form a Hilbert transform pair.

Several design procedures for DTCWTs had been presented in [3]∼[8]. In [7], Selesnick had proposed a common-factor design technique based on the maximally flat allpass filters. This method is simple and effective, but the resulting filters have non-linear phase responses. In [4] and [5], Kingsbury introduced Q-shift filters in order to provide the improved symmetry property. Q-shift filters are required to have linear phase responses. The design technique proposed in [4] and [5] was based on the optimization of a set of rotations θ_i in the polyphase structure, but this is a highly non-linear problem and only works well for relatively short filters. In [8], Kingsbury had proposed an alternative technique for optimizing Q-shift filters, which works effectively for filters of length up to 50 or more taps. This method was based on the minimization of energy of $H_{L2}(z)$ in $[\frac{\pi}{3}, \pi]$, instead of the approximation of group delay for $H_0(z)$. In [12], we have proposed a design method of Q-shift filters with improved vanishing moments, where a set of equations is derived directly from the flatness condition of group delay, vanishing moments and orthogonality condition, and then linearized to obtain the filter coefficients. However, the proposed iterative procedure needs a good initial solution to converge to the optimal solution, particularly if the filter is of higher degree.

In this paper, we propose a new method for designing Q-shift filters with flat group delay responses. We make use of the transfer function proposed by Gopinath in [2] to satisfy the specified degree of flatness for the group delay response at $\omega = 0$ and the specified number of zeros at $z = -1$. Therefore, the design is reduced to how to force the filter to satisfy the condition of orthogonality only. We can derive a set of non-linear equations from the condition of orthogonality. The number of equations is less than half that in the conventional method [12]. Moreover, we linearize the non-linear problem and use an iterative procedure to obtain the filter coefficients. As a result, the Q-shift filters have flat group delay responses and the specified number of vanishing moments. Finally, some examples are presented to demonstrate the effectiveness of the proposed design method.

II. Q-shift Filters for DTCWTs

It is well-known that the DTCWT employs two real DWTs; the first DWT gives the real part of DTCWT and the second DWT is the imaginary part. The second wavelet basis is required to be the Hilbert transform of the first wavelet basis.

Let $\phi_H(t), \phi_G(t)$ and $\psi_H(t), \psi_G(t)$ be the scaling and wavelet functions of two DWTs, respectively. It has been proven in [6], [9] and [10] that two wavelet functions $\psi_H(t)$ and $\psi_G(t)$ form a Hilbert transform pair;

$$\psi_G(t) = \mathcal{H}\{\psi_H(t)\}, \tag{1}$$

that is

$$\Psi_G(\omega) = \begin{cases} -j\Psi_H(\omega) & (\omega > 0) \\ \ \ j\Psi_H(\omega) & (\omega < 0) \end{cases}, \tag{2}$$

if and only if two scaling lowpass filters satisfy

$$G(e^{j\omega}) = H(e^{j\omega})e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{3}$$

where $\Psi_H(\omega)$ and $\Psi_G(\omega)$ are the Fourier transforms of $\psi_H(t), \psi_G(t)$, respectively. This is the so-called half-sample delay condition between two scaling lowpass filters $H(z)$ and $G(z)$. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. Eq.(3) is the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair [10].

In [4] and [5], Kingsbury had proposed Q-shift filters in order to provide the improved orthogonality and symmetry properties. One scaling lowpass filter is chosen to be the time reverse of another filter;

$$G(z) = z^{-N} H(z^{-1}), \tag{4}$$

where $H(z)$ is FIR filter of degree N. Its transfer function is given by

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n}, \tag{5}$$

where $h(n)$ are real filter coefficients and N is an odd number.

Q-shift filters are required to have linear phase responses. That is, the desired phase response of $H(z)$ is

$$\theta_d(\omega) = -(\frac{N}{2} - \frac{1}{4})\omega. \tag{6}$$

Therefore, the phase response of $G(z)$ will be $-(\frac{N}{2} + \frac{1}{4})\omega$, and then two scaling lowpass filters satisfy the half-sample delay condition in Eq.(3).

In addition to the phase condition given in Eq.(6), $H(z)$ is also required to satisfy the conditions of regularity and orthonormality of wavelets. From the viewpoint of regularity, $H(z)$ must have K zeros at $z = -1$;

$$H(z) = Q(z)(1 + z^{-1})^K. \tag{7}$$

When the maximum K is chosen, the maximum number of vanishing moments can be obtained.

Moreover, the condition of orthonormality for $H(z)$ is given by

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2, \tag{8}$$

which means that the product filter $P(z) = H(z)H(z^{-1})$ must satisfy

$$p(2n) = \delta(n) = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases}, \tag{9}$$

where $p(n) = p(-n)$ is the impulse response of $P(z)$.

III. Design of Q-shift Filters

In this section, we discuss the design of Q-shift filters with flat group delay responses. The group delay response $\tau(\omega)$ is required to have the specified degree of flatness at $\omega = 0$;

$$\begin{cases} \tau(0) = \tau_0 \\ \left.\dfrac{\partial^{2k}\tau(\omega)}{\partial\omega^{2k}}\right|_{\omega=0} = 0 \quad (k = 1, 2, \cdots, L-1) \end{cases}, \tag{10}$$

where $\tau_0 = N/2 - 1/4$ and $L\,(> 0)$ is a parameter that controls the degree of flatness.

It is known in [1] that the transfer function satisfying the condition of flatness in Eq.(10) can be given by

$$F_{L,\tau_0}(z) = \frac{1}{\beta}[1 + \sum_{n=1}^{L}(-1)^n \binom{L}{n} \prod_{i=0}^{L-1} \frac{i - 2\tau_0}{i + n - 2\tau_0} z^{-n}], \tag{11}$$

where the real coefficient $\beta \neq 0$ can be arbitrarily chosen. To ensure $F_{L,\tau_0}(1) = 1$, we choose

$$\beta = 1 + \sum_{n=1}^{L}(-1)^n \binom{L}{n} \prod_{i=0}^{L-1} \frac{i - 2\tau_0}{i + n - 2\tau_0}. \tag{12}$$

By using Eq.(11), Gopinath had proven in *Lemma 2* of [2] that the transfer function satisfying the condition of flatness in Eq.(10) and having K zeros at $z = -1$ is of the form

$$H(z) = \sum_{k=0}^{N-L-K} \alpha_k z^{-k} (\frac{1 + z^{-1}}{2})^K F_{L,\tau_0 - k - \frac{K}{2}}(z), \tag{13}$$

where α_k are the real filter coefficients. By using the transfer function in Eq.(13) to design Q-shift filters, thus, we need only to consider the condition of orthonormality in Eq.(8).

By substituting $H(z)$ in Eq.(13) into Eq.(8), we have the product filter $P(z) = H(z)H(z^{-1})$ as

$$P(z) = \sum_{k_1=0}^{N-L-K} \sum_{k_2=0}^{N-L-K} \alpha_{k_1}\alpha_{k_2} D_{k_1,k_2}(z), \tag{14}$$

where

$$D_{k_1,k_2}(z) = z^{-k_1+k_2}(\frac{1 + z^{-1}}{2})^K (\frac{1 + z}{2})^K \times F_{L,\tau_0 - k_1 - \frac{K}{2}}(z) F_{L,\tau_0 - k_2 - \frac{K}{2}}(z^{-1}). \tag{15}$$

Let $d_{k_1,k_2}(n)$ is the impulse response of $D_{k_1,k_2}(z)$, where $d_{k_1,k_2}(n) = 0$ for $n < -(K + L) + k_1 - k_2$ and $n > (K + L) + k_1 - k_2$. We can derive a set of equations as follows;

$$\sum_{k_1=0}^{N-L-K} \sum_{k_2=0}^{N-L-K} \alpha_{k_1}\alpha_{k_2} d_{k_1,k_2}(2n) = \delta(n). \tag{16}$$

It is clear that there are $(N + 1)/2$ equations with respect to $(N - L - K + 1)$ unknowns α_k. If $N = 2(L + K) - 1$, then we can obtain the filter of minimal degree for given L and K, which corresponds to the maximal K ($K_{max} = (N+1)/2 - L$) for given N and L. Therefore, we consider the case of $N = 2(L + K) - 1$ to obtain the maximum number of vanishing moments.

Let $\tilde{\alpha}_k = \alpha_k / \alpha_M$. From Eq.(16), we have

$$\alpha_M^2 = \frac{1}{\sum_{k_1=0}^{L+K-1} \sum_{k_2=0}^{L+K-1} \tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2} d_{k_1,k_2}(0)}, \tag{17}$$

and

$$\sum_{k_1=0}^{L+K-1} \sum_{k_2=0}^{L+K-1} \tilde{\alpha}_{k_1}\tilde{\alpha}_{k_2} d_{k_1,k_2}(2n) = 0 \qquad (n > 0). \tag{18}$$

Note that $\tilde{\alpha}_M = 1$. Thus, we firstly solve Eq.(18) to obtain $\tilde{\alpha}_k$, and then compute α_M by Eq.(17) to get $\alpha_k = \alpha_M \tilde{\alpha}_k$. M can be arbitrarily chosen between $0 \leq M \leq L + K - 1$. In this paper, we choose $M = \lfloor \frac{L+K}{2} \rfloor$, where $\lfloor x \rfloor$ means the largest integer not greater than x. It should be noted that the number of equations in Eq.(18) is $(N - 1)/2 = L + K - 1$, and is less than half that in [12].

It is seen that Eq.(18) is a set of quadratic constraints on the filter coefficients $\tilde{\alpha}_k$. It is difficult to solve the non-linear problem in Eq.(18), particularly if the filter is of higher degree. We have used the function *solve()* in the Symbolic Math Toolbox of MATLAB to solve Eq.(18), but it only works well for the filters of degree $N \leq 9$.

IV. An Iterative Procedure

In this section, we firstly linearize the non-linear problem in Eq.(18), and then use an iterative procedure to obtain a set of filter coefficients $\tilde{\alpha}_k$.

Let $\tilde{\alpha}_k^{(i)}$ be the filter coefficients at ith iteration, and is given by

$$\tilde{\alpha}_k^{(i)} = \tilde{\alpha}_k^{(i-1)} + \Delta\tilde{\alpha}_k^{(i)} \qquad (k \neq M). \tag{19}$$

Then, Eq.(18) becomes

$$\sum_{k_1=0}^{L+K-1} \sum_{k_2=0}^{L+K-1} [\tilde{\alpha}_{k_1}^{(i-1)}\tilde{\alpha}_{k_2}^{(i-1)} + \tilde{\alpha}_{k_1}^{(i-1)}\Delta\tilde{\alpha}_{k_2}^{(i)} + \tilde{\alpha}_{k_2}^{(i-1)}\Delta\tilde{\alpha}_{k_1}^{(i)} + \Delta\tilde{\alpha}_{k_1}^{(i)}\Delta\tilde{\alpha}_{k_2}^{(i)}]d_{k_1,k_2}(2n) = 0 \quad (n > 0). \tag{20}$$

If $\Delta\tilde{\alpha}_k^{(i)}$ is assumed to become small as i increases, $\Delta\tilde{\alpha}_{k_1}^{(i)}\Delta\tilde{\alpha}_{k_2}^{(i)}$ can be neglected. Thus, we have

$$\sum_{\substack{k_1=0\\ \neq M}}^{L+K-1} \{ \sum_{k_2=0}^{L+K-1} [d_{k_1,k_2}(2n) + d_{k_2,k_1}(2n)]\tilde{\alpha}_{k_2}^{(i-1)} \}\Delta\tilde{\alpha}_{k_1}^{(i)} = -\sum_{k_1=0}^{L+K-1} \sum_{k_2=0}^{L+K-1} \tilde{\alpha}_{k_1}^{(i-1)}\tilde{\alpha}_{k_2}^{(i-1)} d_{k_1,k_2}(2n) \quad (n > 0). \tag{21}$$

Note that $\Delta\tilde{\alpha}_M^{(i)} = 0$, since $\tilde{\alpha}_M^{(i)} = 1$. Therefore, we can get $\Delta\tilde{\alpha}_k^{(i)}$ by solving this system of linear equations in Eq.(21), if $\tilde{\alpha}_k^{(i-1)}$ are previously known. The filter coefficients $\tilde{\alpha}_k^{(i)}$ are updated by $\Delta\tilde{\alpha}_k^{(i)}$ in Eq.(19).

To converge to the optimal solution, a set of good initial coefficients $\tilde{\alpha}_k^{(0)}$ is needed. In this paper, we use the filter coefficients of the obtained Q-shift filters of degree $N-2$ or the same degree N but with different K, L as a set of initial coefficients $\tilde{\alpha}_k^{(0)}$.

V. Design Examples

In this section, we present two design examples and compare it with the Q-shift filter designed by Kingsbury in [8] to demonstrate the effectiveness of the proposed design method.

Example 1 We have designed the Q-shift filters with $N = 13$ by using the proposed design method. Firstly, $K = 3$ and $L = 4$ has been chosen, and then the resulting magnitude and group delay responses are shown in solid line in Fig.1 and Fig.2, respectively. For comparison, the magnitude and group delay responses of the Q-shift filters with $K = 2, L = 5$ and designed by Kingsbury are shown in Fig.1 and Fig.2 also. It is seen in Fig.1 that the magnitude responses become more sharp as K increases. It is noted that the Q-shift filter designed by Kingsbury has a sharp magnitude response, but only one zero at $z = -1$, which means the wavelet has only one vanishing moment. It is clear in Fig.2 that the group delay responses become more flat as L increases, and it is better than the Q-shift filter designed by Kingsbury.

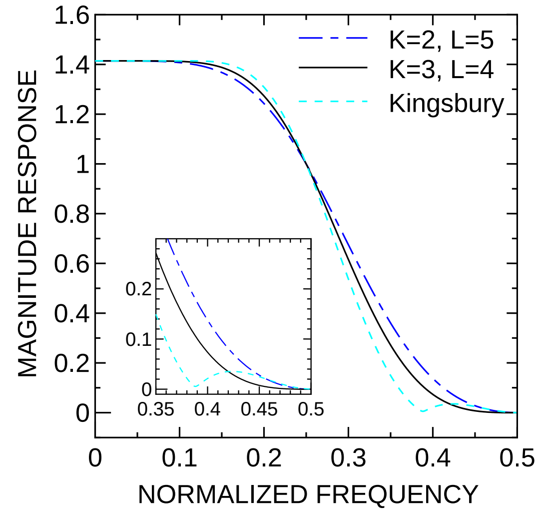

Fig. 1. Magnitude responses of Q-shift filters with $N = 13$.

It is clear in Eq.(3) that $G(e^{j\omega})$ needs to be approximated to $H(e^{j\omega})e^{-j\frac{\omega}{2}}$. For the purpose of comparison, we define the error function $E(\omega)$ as

$$E(\omega) = G(e^{j\omega}) - H(e^{j\omega})e^{-j\frac{\omega}{2}}. \tag{22}$$

The magnitude $|E(\omega)|$ of these Q-shift filters are shown in Fig.3.

Example 2 We have designed Q-shift filters of degree $N = 35$ with $K = 16, 15, 14$ and $L = 2, 3, 4$. The resulting magnitude and group delay responses are shown Fig.4 and Fig.5, respectively. It is seen in Fig.4 and Fig.5 that the magnitude responses are more sharp as K increases, while the group delay responses are more flat as L increases.

VI. Conclusion

In this paper, we have proposed a new method for designing Q-shift filters with flat group delay responses. We have used the transfer function proposed by Gopinath to satisfy both the specified degree of flatness for the group delay response at $\omega = 0$ and the specified number of zeros at $z = -1$ for the regularity of wavelets. Therefore, the design problem can be reduced to how the condition of orthonormality to be satisfied. We have derived a set of non-linear equations from the condition of orthonormality, and then linearized the non-linear problem. Since the number of equations is less than half that in the conventional method [12], we can obtain efficiently the filter coefficients by using an iterative procedure. As a result, the obtained Q-shift filters have flat group delay responses and the specified number of vanishing moments. Finally, some examples are presented to demonstrate the effectiveness of the design method proposed in this paper.

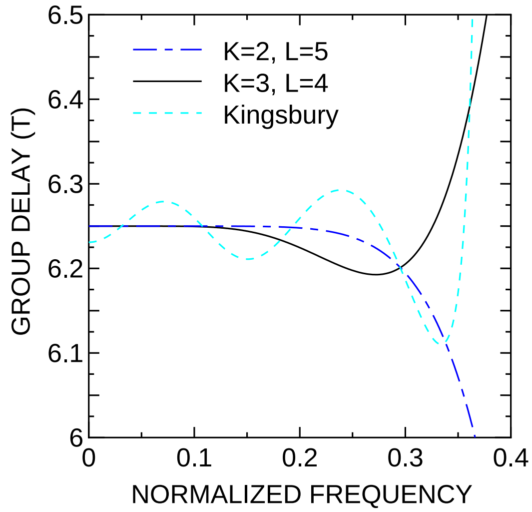

Fig. 2. Group delay responses of Q-shift filters with $N = 13$.

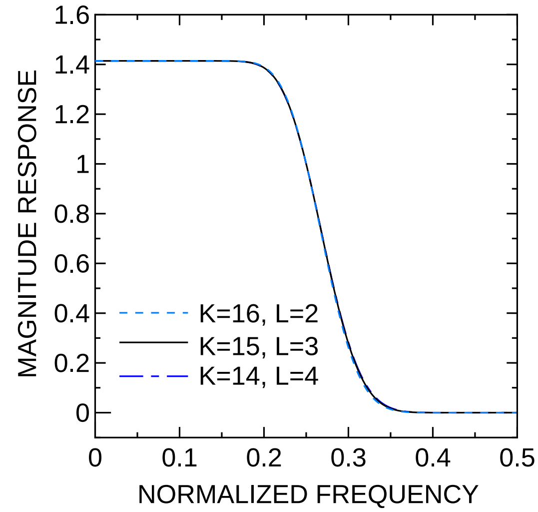

Fig. 4. Magnitude responses of Q-shift filters with $N = 35$.

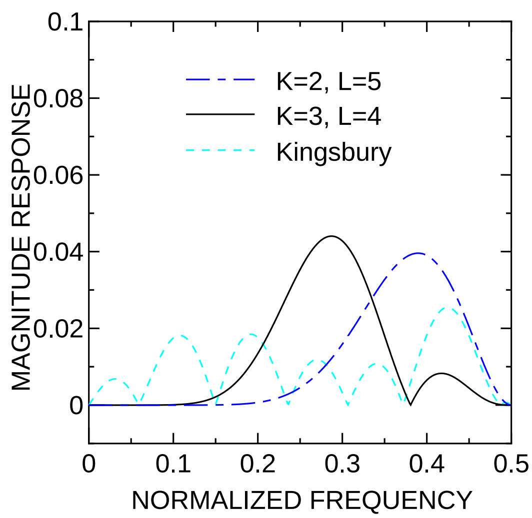

Fig. 3. Magnitude responses of error functions $E(\omega)$ of Q-shift filters.

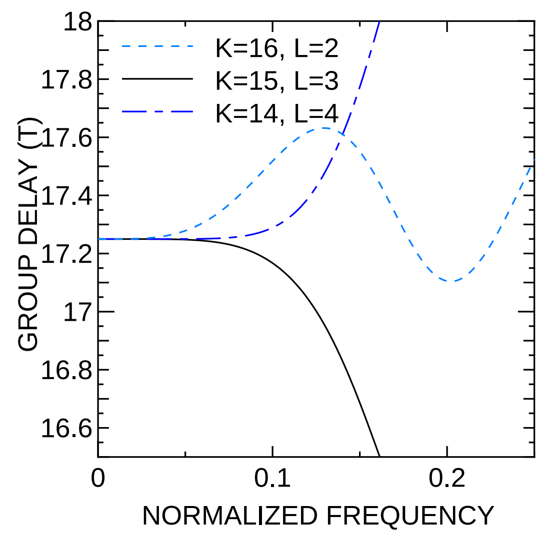

Fig. 5. Group delay responses of Q-shift filters with $N = 35$.

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