DESIGN OF SYMMETRIC ORTHONORMAL WAVELET FILTERS USING A SINGLE COMPLEX ALLPASS FILTER

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ABSTRACT

This paper gives a new class of real-valued symmetric orthonormal wavelet filters by using a single complex allpass filter. Firstly, the conditions that the complex allpass filter has to satisfy are derived from the symmetric and orthonormal conditions of wavelets. Then, from the viewpoint of the wavelet regularity, a new method is proposed for designing the proposed symmetric orthonormal wavelet filters with the maximally flatness. In the proposed method, the maximally flat solutions can be easily obtained by solving a set of linear equations only.

1. INTRODUCTION

It is well known [1],[3],[4] that orthonormal wavelets can be generated by two-band paraunitary filter banks. Symmetric orthonormal wavelets require all filters in two-band paraunitary filter banks to have an exact linear phase response. It is widely appreciated that the only FIR solution that produces a real-valued symmetric orthonormal wavelet basis is the Haar solution, which is not continuous. To obtain a real-valued symmetric orthonormal wavelet basis with more regularity than the Haar solution, Herley and Vetterli had proposed a class of IIR solutions in [4]. In [4], Herley and Vetterli discussed two cases: half sample symmetric (HSS) and whole sample symmetric (WSS). In the HSS case, the scaling and wavelet functions are symmetric and antisymmetric, respectively, while in the WSS case, both the scaling and wavelet functions are symmetric. Herley and Vetterli had shown in [4] that the HSS wavelet filters can be constructed by using real allpass filters. However, the WSS wavelet filters is not as easy as in the HSS case, and Herley and Vetterli showed one example only.

In this paper, we will discuss the WSS case and give a new class of real-valued symmetric orthonormal wavelet filters by using a single complex allpass filter. Firstly, we construct a two-band paraunitary filter bank by using a single complex allpass filter, and then derive the conditions that the complex allpass filter has to satisfy from the symmetric and orthonormal conditions of wavelets. Secondly, from the viewpoint of the wavelet regularity, we propose a new method for designing the proposed symmetric orthonormal wavelet filters with the maximally flatness. In the proposed method, the maximally flat solutions can be easily obtained by solving a set of linear equations only. Finally, some design examples are presented to demonstrate the effectiveness of the proposed method.

2. SYMMETRIC ORTHONORMAL IIR WAVELET FILTERS

It is well known [1],[3],[4] that an orthonormal wavelet basis can be generated by a two-band paraunitary filter bank $\{H(z), G(z)\}$, where $H(z)$ is a lowpass filter and $G(z)$ is highpass. The orthonormal condition that $H(z)$ and $G(z)$ have to satisfy is

$$\begin{cases} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1 \\ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 1 \\ H(z)G(z^{-1}) + H(-z)G(-z^{-1}) = 0 \end{cases} . \quad (1)$$

When the orthonormal wavelet basis is required to be symmetric, $H(z)$ and $G(z)$ must have an exact linear phase response. In [4], Herley and Vetterli had given a class of linear phase IIR solutions by using real allpass filters, which is the HSS case, i.e., the numerator degree of $H(z)$ and $G(z)$ is odd. Now, we will consider the WSS case, i.e., the numerator degree of $H(z)$ and $G(z)$ must be even. According to [2], we construct $H(z)$ and $G(z)$ by using a single complex allpass filter as follows;

$$\begin{cases} H(z) = \frac{1}{2}\{A(z) + \tilde{A}(z)\} \\ G(z) = \frac{z^{-1}}{2j}\{A(z) - \tilde{A}(z)\} \end{cases} , \quad (2)$$

where $A(z)$ is a complex allpass filter, and $\tilde{A}(z)$ has a set of filter coefficients that are complex conjugate with ones of $A(z)$. One can verify that $H(z)$ and $G(z)$ have a set of real-valued filter coefficients and the numerator degree is even. To satisfy the orthonormal conditions in Eq.(1), $A(z)$ must satisfy [2]

$$A(z) = \pm j\tilde{A}(-z), \tag{3}$$

which means that if α is a pole of $A(z)$, then $-\alpha^*$ is also a pole of $A(z)$. Consequently, $A(z)$ has a pair of poles $(\alpha, -\alpha^*)$ and/or one pole $j\beta$, where α is complex, β is real, and α^* denotes the complex conjugate of α. To force $H(z)$ and $G(z)$ to have an exact linear phase response, $A(z)$ must satisfy also

$$A(z) = \frac{1}{\tilde{A}(z)}. \tag{4}$$

It should be noted that $A(z)$ and $\tilde{A}(z)$ satisfy the following relation;

$$A(z) = \frac{1}{\tilde{A}(z^{-1})}. \tag{5}$$

Thus, the condition of Eq.(4) becomes

$$A(z) = A(z^{-1}), \tag{6}$$

which means that if α is a pole of $A(z)$, then $1/\alpha$ is also a pole of $A(z)$. Therefore, $A(z)$ has a quadruplet of poles $(\alpha, 1/\alpha, -\alpha^*, -1/\alpha^*)$ and/or a pair of poles $(j\beta, 1/j\beta)$, and can be expressed as

$$A(z) = \eta z^{-N} \prod_{k=1}^{N_1} \frac{(1+j\beta_k z)(1-j\beta_k^{-1} z)}{(1-j\beta_k z^{-1})(1+j\beta_k^{-1} z^{-1})}$$
$$\prod_{k=1}^{N_2} \frac{(1-\alpha_k^* z)(1-\frac{z}{\alpha_k^*})(1+\alpha_k z)(1+\frac{z}{\alpha_k})}{(1-\alpha_k z^{-1})(1-\frac{z^{-1}}{\alpha_k})(1+\alpha_k^* z^{-1})(1+\frac{z^{-1}}{\alpha_k^*})}, \tag{7}$$

where $N = 2N_1 + 4N_2$, and $\eta = \pm\exp\{\pm j\pi/4\}$. By expanding Eq.(7), we get

$$A(z) = \eta z^{-N} \frac{a_0 + ja_1 z + a_2 z^2 + \cdots}{a_0 - ja_1 z^{-1} + a_2 z^{-2} + \cdots}$$
$$\frac{\cdots + a_2 z^{N-2} + ja_1 z^{N-1} + a_0 z^N}{\cdots + a_2 z^{-N+2} - ja_1 z^{-N+1} + a_0 z^{-N}}, \tag{8}$$

where a_n are real filter coefficients, and $a_0 = 1$. Therefore, to use the transfer function of $A(z)$ in Eq.(8) implies that the symmetric and orthonormal conditions have been satisfied. In the following section, we will consider how to design $H(z)$ and $G(z)$, i.e., $A(z)$.

3. DESIGN OF WAVELET FILTERS WITH MAXIMALLY FLATNESS

In this section, we describe the design of the proposed symmetric orthonormal IIR wavelet filters with the maximally flatness, from the viewpoint of the wavelet regularity [3]~[5].

3.1. Frequency Response

Let $\theta(\omega)$ be the phase response of $A(z)$. From Eq.(8), we have

$$\theta(\omega) = \theta_0 + 2\varphi(\omega), \tag{9}$$

where $\theta_0 = \pm\pi/4$ or $\pm 3\pi/4$, and when $N/2$ is even,

$$\varphi(\omega) = \tan^{-1} \frac{\sum_{n=0}^{N/4-1} a_{2n+1} \cos(\frac{N}{2} - 2n - 1)\omega}{\frac{a_{N/2}}{2} + \sum_{n=0}^{N/4-1} a_{2n} \cos(\frac{N}{2} - 2n)\omega} = \tan^{-1} \frac{N(\omega)}{D(\omega)}, \tag{10}$$

and when $N/2$ is odd,

$$\varphi(\omega) = \tan^{-1} \frac{\frac{a_{N/2}}{2} + \sum_{n=0}^{(N-6)/4} a_{2n+1} \cos(\frac{N}{2} - 2n - 1)\omega}{\sum_{n=0}^{(N-2)/4} a_{2n} \cos(\frac{N}{2} - 2n)\omega} = \tan^{-1} \frac{N(\omega)}{D(\omega)}. \tag{11}$$

From Eq.(2), we have

$$\begin{cases} H(e^{j\omega}) = \cos\theta(\omega) \\ G(e^{j\omega}) = e^{-j\omega} \sin\theta(\omega) \end{cases}. \tag{12}$$

It is clear that both $H(z)$ and $G(z)$ have an exact linear phase response, and their magnitude responses satisfy the following power-complementary relation;

$$|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1, \tag{13}$$

which means that either $H(z)$ or $G(z)$ needs to be designed. In the following, we will consider the design of $G(z)$ for convenience.

3.2. Desired Phase Response

$H(z)$ and $G(z)$ are required to be a pair of lowpass and highpass filters. The desired magnitude responses are

$$|H_d(e^{j\omega})| = \begin{cases} 1 & (0 \le \omega \le \omega_p) \\ 0 & (\omega_s \le \omega \le \pi) \end{cases}, \tag{14}$$

$$|G_d(e^{j\omega})| = \begin{cases} 0 & (0 \le \omega \le \omega_p) \\ 1 & (\omega_s \le \omega \le \pi) \end{cases}, \qquad (15)$$

where ω_p and ω_s are the cutoff frequencies of passband and stopband of $H(z)$, respectively, and $\omega_p + \omega_s = \pi$. Therefore, from Eq.(12), the desired phase response of $A(z)$ is

$$\theta_d(\omega) = \begin{cases} 0 & (0 \le \omega \le \omega_p) \\ \pm\frac{\pi}{2} & (\omega_s \le \omega \le \pi) \end{cases}, \qquad (16)$$

which means from Eq.(9) that the desired response of $\varphi(\omega)$ is

$$\varphi_d(\omega) = \begin{cases} -\frac{\theta_0}{2} & (0 \le \omega \le \omega_p) \\ \pm\frac{\pi}{4} - \frac{\theta_0}{2} & (\omega_s \le \omega \le \pi) \end{cases}. \qquad (17)$$

When $N/2$ is even, it can be seen from Eq.(10) that $\varphi(\pi/2) = 0$ and $\varphi(\omega) = -\varphi(\pi - \omega)$, then we must choose $\theta_0 = \pm\pi/4$. When $N/2$ is odd, it is clear from Eq.(11) that $\varphi(\pi/2) = \pm\pi/2$ and $\varphi(\omega) = \pm\pi - \varphi(\pi - \omega)$, thus $\theta_0 = \pm 3\pi/4$. Therefore, the design problem of $H(z)$ and $G(z)$ will become the approximation of $\varphi(\omega)$. Due to the symmetry of $\varphi(\omega)$, $\varphi(\omega)$ is needed to approximate only in the passband.

3.3. Formulation

From the regularity condition of wavelets, $H(z)$ and $G(z)$ are required to meet the following maximally flatness constraint [3],[4];

$$\left.\frac{\partial^i |H(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=\pi} = 0 \qquad (i = 0, 1, \cdots, N-1), \quad (18)$$

$$\left.\frac{\partial^i |G(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \qquad (i = 0, 1, \cdots, N-1), \quad (19)$$

which implies that $H(z)$ and $G(z)$ contain N zeros located at $z = -1$ and $z = 1$, respectively. For convenience, we consider the design of $G(z)$. Directly using the condition of Eq.(19) will result in a set of nonlinear equations to be solved, which is difficult when N is large. To avoid this problem, we decompose $|G(e^{j\omega})|$ as

$$\begin{aligned} |G(e^{j\omega})| &= \sin\theta(\omega) = 2\sin\frac{\theta(\omega)}{2}\cos\frac{\theta(\omega)}{2} \\ &= 2|G_1(e^{j\omega})||G_2(e^{j\omega})|, \end{aligned} \qquad (20)$$

where

$$\begin{aligned} |G_1(e^{j\omega})| &= \sin\frac{\theta_0}{2}\cos\varphi(\omega) + \cos\frac{\theta_0}{2}\sin\varphi(\omega) \\ &= \frac{\sin\frac{\theta_0}{2}D(\omega) + \cos\frac{\theta_0}{2}N(\omega)}{\{N(\omega)^2 + D(\omega)^2\}^{\frac{1}{2}}}, \end{aligned} \qquad (21)$$

$$\begin{aligned} |G_2(e^{j\omega})| &= \cos\frac{\theta_0}{2}\cos\varphi(\omega) - \sin\frac{\theta_0}{2}\sin\varphi(\omega) \\ &= \frac{\cos\frac{\theta_0}{2}D(\omega) - \sin\frac{\theta_0}{2}N(\omega)}{\{N(\omega)^2 + D(\omega)^2\}^{\frac{1}{2}}}. \end{aligned} \qquad (22)$$

By differentiating Eq.(20), we have

$$\frac{\partial^i |G(e^{j\omega})|}{\partial \omega^i} = 2\sum_{k=0}^{i} \frac{i!}{k!(i-k)!} \frac{\partial^k |G_1(e^{j\omega})|}{\partial \omega^k} \frac{\partial^{i-k} |G_2(e^{j\omega})|}{\partial \omega^{i-k}}. \qquad (23)$$

Therefore, due to $|G_2(1)| = 1$ ideally, the condition of Eq.(19) is equivalent to

$$\left.\frac{\partial^i |G_1(e^{j\omega})|}{\partial \omega^i}\right|_{\omega=0} = 0 \qquad (i = 0, 1, \cdots, N-1). \quad (24)$$

Similarly, from Eq.(21), the condition of Eq.(24) can be reduced to

$$\left.\frac{\partial^i \{\sin\frac{\theta_0}{2}D(\omega) + \cos\frac{\theta_0}{2}N(\omega)\}}{\partial \omega^i}\right|_{\omega=0} = 0, \qquad (25)$$

for $i = 0, 1, \cdots, N-1$. When $N/2$ is even, we substitute Eq.(10) into Eq.(25) and get

$$\begin{cases} a_{N/2}\frac{\sin\frac{\theta_0}{2}}{2} + \sum_{n=0}^{N/4-1}\{a_{2n}\sin\frac{\theta_0}{2} + a_{2n+1}\cos\frac{\theta_0}{2}\} = 0 \\ \sum_{n=0}^{N/4-1}\{a_{2n}(\frac{N}{2} - 2n)^{2i}\sin\frac{\theta_0}{2} + a_{2n+1}(\frac{N}{2} - 2n - 1)^{2i} \\ \qquad \cos\frac{\theta_0}{2}\} = 0 \qquad (i = 1, 2, \cdots, \frac{N}{2} - 1) \end{cases} \qquad (26)$$

which is a set of linear equations. Due to $a_0 = 1$, we can solve Eq.(26) to obtain a set of filter coefficients. When $N/2$ is odd, by substituting Eq.(11) into Eq.(25), we can get a set of linear equations similarly, which is omitted here. Therefore, the maximally flat solutions can be easily obtained by solving the above linear equations only.

4. DESIGN EXAMPLES

In this section, we have designed the symmetric orthonormal IIR wavelet filters with the maximally flatness by using the proposed method. The phase responses of $A(z)$ with $N = 4, 6, 8$ are shown in Fig.1, and the magnitude responses of $H(z)$ and $G(z)$ are shown in Fig.2, respectively. The scaling and wavelet functions generated by the wavelet filters with $N = 6$ are shown in Fig.3 and Fig.4, respectively. It can be seen in Fig.3 and Fig.4 that both the scaling and wavelet functions are symmetric.

5. CONCLUSION

In this paper, we have given a new class of real-valued symmetric orthonormal IIR wavelet filters by using a single complex allpass filter. Firstly, from the symmetric and orthonormal conditions of wavelets, we have derived the conditions that the complex allpass filter has to satisfy. Secondly, from the viewpoint of the wavelet regularity, we have proposed a new method for designing the proposed symmetric orthonormal wavelet filters with the maximally flatness. In the proposed method, the maximally flat solutions can be easily obtained by solving a set of linear equations only. Finally, some design examples have been presented to demonstrate the effectiveness of the proposed method.

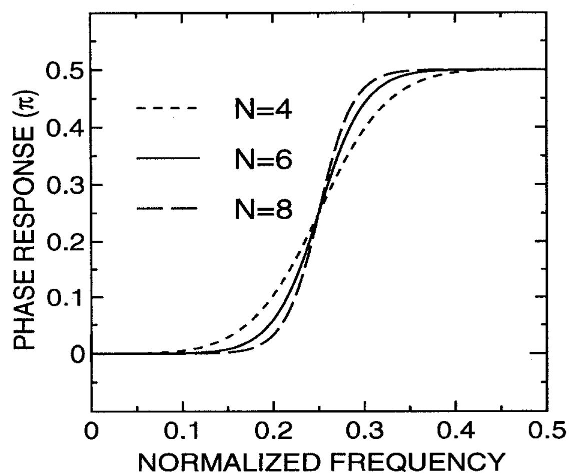

Fig.1 Phase responses of $A(z)$.

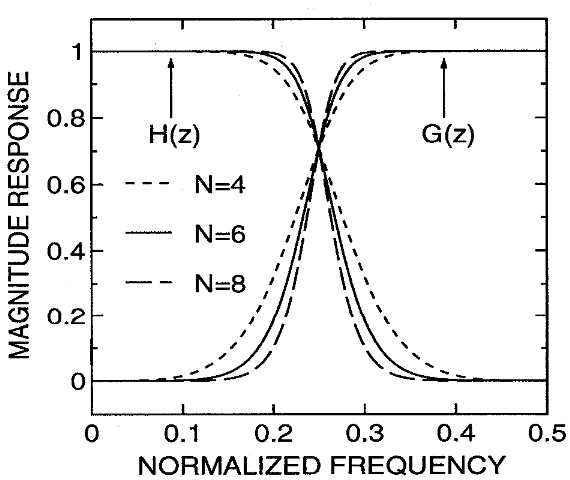

Fig.2 Magnitude responses of $H(z)$ and $G(z)$.

6. REFERENCES

[1] P.P.Vaidyanathan, "Multirate Systems and Filter Banks", Englewood Cliffs, NJ: Prentice Hall, 1993.

[2] P.P.Vaidyanathan, P.A.Regalia and S.K.Mitra, "Design of doubly complementary IIR digital filters using a single complex allpass filter, with multirate applications", IEEE Trans. Circuits & Syst., vol.CAS-34, no.4, pp.378-389, April 1987.

[3] M.Vetterli and C.Herley, "Wavelets and filter banks: Theory and Design", IEEE Trans. Signal Processing, vol.40, no.9, pp.2207–2232, Sep. 1992.

[4] C.Herley and M.Vetterli, "Wavelets and recursive filter banks", IEEE Trans. Signal Processing, vol.41, no.8, pp.2536–2556, Aug. 1993.

[5] I.W.Selesnick, "Formulas for orthogonal IIR wavelet filters", IEEE Trans. Signal Processing, vol.46, no.4, pp.1138–1141, April 1998.

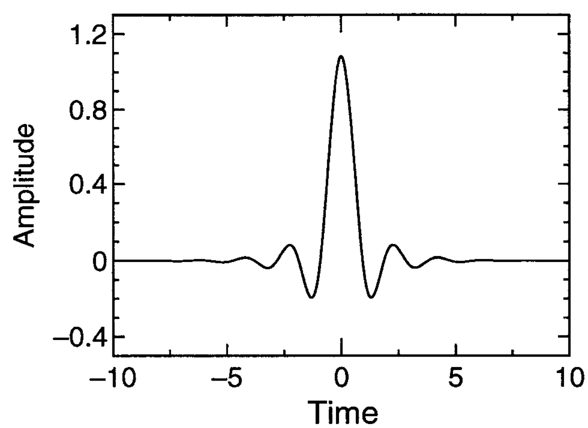

Fig.3 Scaling function.

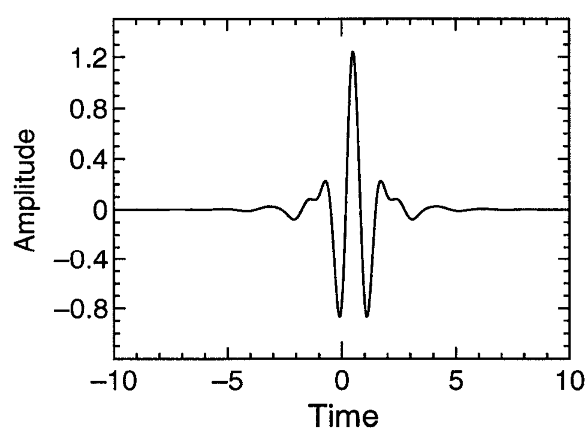

Fig.4 Wavelet function.