Design of Hilbert Transform Pairs of Orthonormal Wavelet Bases Using IIR Filters

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***Abstract*—Conventionally, FIR filters have been often used to design the dual tree complex wavelet transforms (DT$\mathcal{C}$WTs), where two real orthonormal wavelet bases form a Hilbert transform pair, whereas IIR filters are seldom used, although they require a lower computational complexity than FIR filters. In this paper, a new class of Hilbert transform pairs of orthonormal wavelet bases is proposed by using general IIR filters. To obtain the maximum number of vanishing moments, the conventional design methods located as many zeros as possible at $z = -1$. This paper proposes a new design method for DT$\mathcal{C}$WTs by locating a specified number of zeros at $z = -1$ and minimizing the stopband error. The proposed method uses the well-known Remez exchange algorithm to approximate an equiripple magnitude response in the stopband. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem. Furthermore, the optimal solution is attained through a few iterations. The advantage of the proposed method is that the number of zeros at $z = -1$ can be specified arbitrarily and an improved frequency selectivity can be obtained.**

Keywords: **Dual tree complex wavelet transform, Orthonormal wavelet base, Hilbert transform pair, IIR digital filter, Remez exchange algorithm**

I. Introduction

The dual tree complex wavelet transforms (DT$\mathcal{C}$WTs) have been proposed and found to be successful in many applications of signal processing and image processing [8]∼[13]. DT$\mathcal{C}$WTs employ two real wavelet transforms, where one wavelet corresponds to the real part of complex wavelet and the other is the imaginary part. Two wavelet bases are required to form a Hilbert transform pair. Thus, DT$\mathcal{C}$WTs are nearly shift invariant and directionally selective in two and higher dimensions. It has been proven in [11], [14] and [15] that the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair is the half-sample delay condition between the corresponding scaling lowpass filters. Several design procedures for the Hilbert transform pairs of wavelet bases have been proposed in [8]∼[12] by using FIR filters, which are corresponding to the compactly supported wavelets. In [12], Selesnick had proposed a class of Hilbert transform pairs of wavelet bases, where the corresponding scaling lowpass filters are constructed by using an allpass filter to meet the half-sample delay condition. This design method is simple and effective. The approximation accuracy of the half-sample delay is controlled only by the allpass filter. Thus, the design problem becomes how the scaling lowpass filters to satisfy the orthonormality (or biorthogonality) condition and the regularity of wavelets. In [12], Selesnick had used the maximally flat allpass filter, and then given a class of FIR orthonormal and biorthogonal solutions, and IIR orthonormal solution, where the scaling lowpass filters have as many zeros at $z = -1$ as possible to obtain the maximum number of vanishing moments of wavelets, resulting in the maximally flat magnitude response of the scaling lowpass filters. However, for the IIR orthonormal solution proposed in [12], the resulting IIR scaling lowpass filters have the numerator and denominator of the (almost) same degree. In [16], a new class of Hilbert transform pairs of orthonormal wavelet bases has been proposed by using general IIR filters, where the degree of the numerator is larger than that of the denominator, but only the maximally flat design has been discussed.

It is known in [2] and [3] that frequency selectivity is a useful property for many applications of signal processing and image processing. However, the maximally flat filters have a poor frequency selectivity in general. In [17], FIR orthonormal solution proposed in [12] has been modified and a new design procedure has been proposed by specifying the numbers of zeros at $z = -1$ and applying the Remez exchange algorithm [4] in the stopband to get an improved frequency selectivity.

In this paper, we propose a new design procedure for DT$\mathcal{C}$WTs using general IIR filters with numerator and denominator of different degree. We restrict ourself to the orthonormal case. That is, we construct a class of Hilbert transform pairs of orthonormal wavelet bases by using general IIR filters. First, we specify the number of zeros at $z = -1$ to obtain the specified number of vanishing moments, and then use the remaining degree of freedom to improve the frequency selectivity. We apply the Remez exchange algorithm in the stopband of scaling lowpass filters to approximate an equiripple magnitude response. Therefore, the design problem can be easily solved based on the eigenvalue problem [6]. The optimal solution is attained through a few iterations. Furthermore, it is shown that FIR and IIR orthonormal solutions proposed in [12], [16] and [17] are only the special cases of our solution proposed in this paper. Finally, some design examples are presented to demonstrate the effectiveness of the proposed design method.

II. Hilbert Transform Pairs of Wavelet Bases

It is known in [1] that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z), G_i(z)\}$, where $i = 1, 2$. Now we assume that $H_i(z)$ and $G_i(z)$ are lowpass and highpass filters, respectively. The orthonormality condition of two-band filter banks $\{H_i(z), G_i(z)\}$ are given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0 \end{cases}. \quad (1)$$

We denote the scaling and wavelet functions by $\phi_i(t), \psi_i(t)$ respectively. Thus, the corresponding dilation and wavelet equations are expressed as

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n) \end{cases}, \quad (2)$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively.

It is known in [11] that two wavelet functions are a Hilbert transform pair,

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \quad (3)$$

that is,

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \quad (4)$$

if and only if two scaling lowpass filters satisfy the following condition;

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \quad (5)$$

where $\Psi_i(\omega)$ is the Fourier transform of $\psi_i(t)$. Eq.(5) is the so-called half-sample delay condition and it is the necessary and sufficient condition for two wavelet functions to form a Hilbert transform pair.

III. Hilbert Transform Pairs of Orthonormal Wavelet Bases Using IIR Filters

The transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)}, \quad (6)$$

where

$$D(z) = 1 + \sum_{n=1}^{L} d(n)z^{-n}, \quad (7)$$

where L is the degree of $A(z)$ and $d(n)$ are real filter coefficients.

In [12], Selesnick has proposed that the scaling lowpass filters $H_1(z)$ and $H_2(z)$ are composed of an allpass filter by

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}) \end{cases}. \quad (8)$$

Since both of scaling lowpass filters have the same component $F(z)$, then we have

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z). \quad (9)$$

It is clear that $H_2(z)$ is expressed as the product of $H_1(z)$ and $A(z)$. Therefore, if we want to get a Hilbert transform pair, the allpass filter $A(z)$ should be an approximate half-sample delay;

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \quad (10)$$

thus the half-sample delay condition in Eq.(5) is achieved approximately, and two wavelet bases form an approximate Hilbert transform pair.

There are many design methods for allpass filters to approximate a fractional delay response, for example, the maximally flat, least squares [5], equiripple approximations [7], and so on. In [12], the maximally flat fractional delay allpass filter was adapted, and $\omega = 0$ was chosen for the point of approximation. However, the approximation error will increase as ω goes away from the point of approximation in the maximally flat approximation. Thus, it will be better if the minimax (Chebyshev) phase approximation of allpass filters is used, e.g., [7].

A. FIR orthonormal solution

Once $A(z)$ is determined, $F(z)$ needs to be designed for $H_1(z)$ and $H_2(z)$. To obtain wavelet bases with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1 + z^{-1})^K. \quad (11)$$

Thus

$$\begin{cases} H_1(z) = Q(z)(1 + z^{-1})^K D(z) \\ H_2(z) = Q(z)(1 + z^{-1})^K z^{-L} D(z^{-1}) \end{cases}. \quad (12)$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(1+z)^K(1+z^{-1})^K D(z)D(z^{-1}). \end{aligned} \quad (13)$$

Let $Q(z)$ be a FIR filter and defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-R}^{R} r(n)z^{-n}, \quad (14)$$

$$S(z) = (z + 2 + z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \quad (15)$$

where $r(n) = r(-n)$ for $1 \le n \le R$ and $s(n) = s(-n)$ for $1 \le n \le L+K$, then we have

$$P(z) = R(z)S(z). \quad (16)$$

We can write the orthonormality condition in Eq.(1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} 1 & (n=0) \\ 0 & (1 \le n \le \dfrac{R+L+K}{2}) \end{cases}, \tag{17}$$

where $I_{min} = \max\{-R, 2n-L-K\}$ and $I_{max} = \min\{R, 2n+L+K\}$. Note that $P(z)$ is a halfband filter, thus the degree of $H_i(z)$ is $M = R+L+K$ and is an odd number. Since $r(n) = r(-n)$, there are $(M+1)/2$ equations with respect to $(R+1)$ unknown coefficients $r(n)$ in Eq.(17). Therefore, it is clear that we can obtain the only solution if $(M+1)/2 = R+1$. In [12], Selesnick had chosen $R = L+K-1$ and obtained the filter of minimal degree for given L and K, which is correspondent to the maximal K ($K_{max} = R-L+1 = (M+1)/2 - L$) for given L and R. Thus the scaling lowpass filters have the maximally flat magnitude response, resulting in the maximum number of vanishing moments. This is the FIR orthonormal solution proposed in [12].

B. IIR orthonormal solution

In general, IIR filters require a lower computational complexity, compared with FIR filters. IIR filters can also be used to construct a class of Hilbert transform pairs of orthonormal wavelet bases. In [12], Selesnick has chosen

$$F(z) = \frac{(1+z^{-1})^K}{C(z^2)}, \tag{18}$$

then

$$\begin{cases} H_1(z) = \dfrac{(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \dfrac{(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)} \end{cases}. \tag{19}$$

Therefore, $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= \frac{(1+z)^K(1+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})}. \end{aligned} \tag{20}$$

Defining

$$B(z) = C(z)C(z^{-1}) = \sum_{n=-B}^{B} b(n)z^{-n}, \tag{21}$$

where $b(n) = b(-n)$ for $1 \le n \le B$. From the orthonormality condition in Eq.(1), we have

$$S(z) + S(-z) = 2B(z^2), \tag{22}$$

thus $B = \lfloor \frac{L+K}{2} \rfloor$ and

$$b(n) = s(2n), \tag{23}$$

where $\lfloor x \rfloor$ means the largest integer not greater than x. This is the IIR orthonormal solution proposed in [12]. It is clear that the numerator and denominator of $H_i(z)$ are of degree $M = L+K$ and $2B = 2\lfloor \frac{L+K}{2} \rfloor$ respectively, which are the (almost) same.

C. General IIR orthonormal solution

Now we consider the case of using general IIR filters with numerator and denominator of different degree. We choose

$$F(z) = \frac{Q(z)(1+z^{-1})^K}{C(z^2)}, \tag{24}$$

thus

$$\begin{cases} H_1(z) = \dfrac{Q(z)(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \dfrac{Q(z)(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)} \end{cases}, \tag{25}$$

where the degree of numerator is not less than the degree of denominator, i.e., $M = L+K+R \ge 2B$. If $M > 2B$, M is an odd number, whereas if $M = 2B$, M is an even number.

We have the product filter $P(z)$ as

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= \frac{Q(z)Q(z^{-1})(1+z)^K(1+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})} \\ &= \frac{R(z)S(z)}{B(z^2)}. \end{aligned} \tag{26}$$

From the orthonormality condition, we have

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} b(n) & (0 \le n \le B) \\ 0 & (B < n \le \dfrac{M}{2}) \end{cases}. \tag{27}$$

Assuming $b(0) = 1$, there are $\lfloor \frac{M}{2} \rfloor + 1$ equations with respect to $(R+B+1)$ unknown coefficients $r(n)$ and $b(n)$ in Eq.(27). Therefore, it is clear that the only solution exists if $\lfloor \frac{M}{2} \rfloor + 1 = R+B+1$, which results in the maximum number of vanishing moments. When $M > 2B$, $R+2B = L+K-1$, since M is odd. If we choose $B = 0$, then $R = L+K-1$, which is correspondent to FIR orthonormal solution in [12]. If we choose $R = 0$, then $M = L+K$. When M is odd, $2B = L+K-1 = M-1$, while if M is even, $2B = L+K = M$. Thus $B = \lfloor \frac{M}{2} \rfloor = \lfloor \frac{L+K}{2} \rfloor$, and it is IIR orthonormal solution in [12]. Therefore, it is clear that FIR and IIR orthonormal solutions proposed in [12] are only the special cases of general IIR orthonormal solutions when $B = 0$ and $R = 0$.

Example 1: We consider a Hilbert transform pair of orthonormal wavelet bases with $K = 4$ and $L = 2$ as in [12]. To obtain the maximum number of vanishing moments, we can choose $\{R, B\} = \{5, 0\}, \{3, 1\}, \{1, 2\}, \{0, 3\}$, where the degree of numerator are $M = 11, 9, 7, 6$ respectively. Note that the filter with $\{R, B\} = \{5, 0\}$ is FIR. We have designed these four filters, and the resulting magnitude responses of $H_i(z)$ are shown in Fig.1. It is seen that IIR filters with $B > 0$ have more sharp magnitude responses than FIR filter with $B = 0$. Their group delays are given in Fig.2, and it is clear that $H_1(z)$ and $H_2(z)$ satisfy the half-sample delay condition. Moreover, the magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ are also shown in Fig.3. It is seen in Fig.3 that the maximum errors of IIR filters are smaller than the conventional FIR filter.

IV. IIR Orthonormal Solution with Improved Frequency Selectivity

It is known in [2] and [3] that frequency selectivity is a useful property for many applications of signal processing and image processing. However, the maximally flat filters have a poor frequency selectivity in general. In the conventional design of DT$\mathcal{C}$WTs, the scaling lowpass filters have as many zeros at $z=-1$ as possible to obtain the maximum number of vanishing moments, resulting in the maximally flat magnitude response. In [17], we have modified FIR orthonormal solution in [12] and proposed a new design procedure to improve the frequency selectivity of the filters.

In this section, we discuss the case of general IIR orthonormal solution, and attempt to improve the frequency selectivity of scaling lowpass filters. We firstly specify the number of zeros at $z=-1$ for $H_i(z)$ from the viewpoint of regularity, and then use the remaining degree of freedom to obtain the best possible frequency selectivity.

We assume $K<K_{max}$, where $K_{max}=\lfloor\frac{M+1}{2}\rfloor+B-L$. Besides satisfying the orthonormality condition in Eq.(27), we want to obtain an equiripple magnitude response in the stopband by using the remaining degree of freedom. The remaining degree of freedom is $K_{max}-K$. Since zeros on the unit circle are complex-conjugate pair except $z=\pm1$, $K_{max}-K$ should be even, i.e., $K_{max}-K=2m$.

Next we apply the Remez exchange algorithm in the stopband $[\omega_s,\pi]$ to get an equrrippple magnitude response, where ω_s is the cutoff frequency of scaling lowpass filters. We assume ω_i $(\omega_s=\omega_0<\omega_1<\cdots<\omega_{2m}<\pi)$ to be a set of extremal frequencies and formulate $P(e^{j\omega})$ as

$$P(e^{j\omega_i})=\frac{R(e^{j\omega_i})S(e^{j\omega_i})}{B(e^{j2\omega_i})}=\frac{1+(-1)^i}{2}\delta, \tag{28}$$

where $\delta>0$ is an error. Note that we force $P(e^{j\omega})\geq 0$ to permit spectral factorization of $R(z)$. From Eq.(28), we have

$$S(e^{j\omega_i})R(e^{j\omega_i})=\frac{1+(-1)^i}{2}\delta B(e^{j2\omega_i}), \tag{29}$$

where

$$\begin{cases} R(e^{j\omega})=r(0)+2\sum_{n=1}^{R}r(n)\cos(n\omega) \\ B(e^{j\omega})=b(0)+2\sum_{n=1}^{B}b(n)\cos(n\omega) \end{cases}. \tag{30}$$

We rewrite Eq.(27) and Eq.(29) in the matrix form as

$$\mathbf{P}\mathbf{x}=\delta\mathbf{Q}\mathbf{x} \tag{31}$$

where $\mathbf{x}=[r(0),r(1),\cdots,r(R),b(0),b(1),\cdots,b(B)]^T$,

$$\mathbf{P}=\begin{bmatrix} s(0) & 2s(1) & 2s(2) & \cdots & -1 & 0 & \cdots \\ s(2) & s(1)+s(3) & s(0)+s(4) & \cdots & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ S(e^{j\omega_0}) & 2S(e^{j\omega_0})\cos(\omega_0) & 2S(e^{j\omega_0})\cos(2\omega_0) & \cdots & 0 & 0 & \cdots \\ S(e^{j\omega_1}) & 2S(e^{j\omega_1})\cos(\omega_1) & 2S(e^{j\omega_1})\cos(2\omega_1) & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ S(e^{j\omega_{2m}}) & 2S(e^{j\omega_{2m}})\cos(\omega_{2m}) & 2S(e^{j\omega_{2m}})\cos(2\omega_{2m}) & \cdots & 0 & 0 & \cdots \end{bmatrix}$$

$$\mathbf{Q}=\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2\cos(2\omega_0) & \cdots & 2\cos(2B\omega_0) \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2\cos(2\omega_{2m}) & \cdots & 2\cos(2B\omega_{2m}) \end{bmatrix}$$

which are of size $(R+B+2)\times(R+B+2)$. Therefore, Eq.(31) is correspondent to a generalized eigenvalue problem.

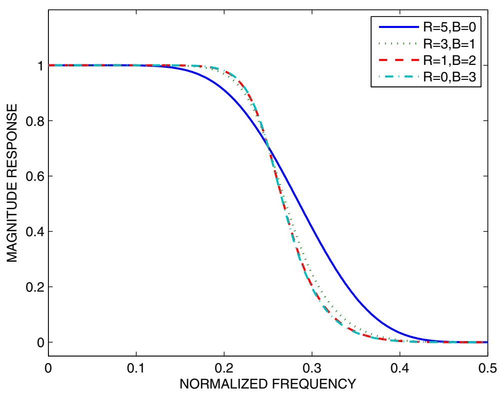

Fig. 1. Magnitude responses of scaling lowpass filters $H_i(z)$ in Example 1.

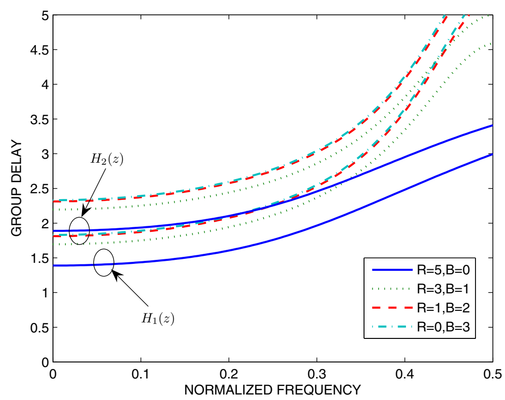

Fig. 2. Group delays of scaling lowpass filters $H_i(z)$ in Example 1.

That is, δ is the eigenvalue and $\mathbf{x}$ is the corresponding eigenvector. From the definition of eigenvalue, it is known that there are not only one eigenvalue. We choose the minimum positive eigenvalue to get the minimum magnitude error δ. Therefore, the corresponding eigenvector x gives a set of filter coefficients $r(n)$ and $b(n)$. However, the initial extremal frequencies may not be the true peak frequencies. We make use of an iteration procedure to find the extremal frequencies to obtain the optimal equiripple magnitude response. The optimal coefficients $r(n)$ and $b(n)$ can be easily obtained through a few iterations. Finally, we use the spectral factorization approach to get $Q(z), C(z)$ from $R(z), B(z)$ to construct scaling lowpass filters $H_1(z), H_2(z)$. The design algorithm is shown as follows.

Design Algorithm {Design of DT$\mathcal{C}$WTs with Improved Frequency Selectivity}

Begin

1) Read M, B, K, L, and the stopband cutoff frequency ω_s.
2) Design $A(z)$ to get $d(n)$, and use Eq.(15) to compute $s(n)$.
3) Select initial extremal frequencies Ω_i ($\omega_s = \Omega_0 < \Omega_1 < \cdots < \Omega_{2m} < \pi$) equally spaced in the stopband.

Repeat

4) Set $\omega_i = \Omega_i$ for $i = 0, 1, \cdots, 2m$.
5) Compute the minimum positive eigenvalue and corresponding eigenvector in Eq.(31) to obtain a set of filter coefficients $r(n), b(n)$.
6) Search the peak frequencies Ω_i ($\omega_s = \Omega_0 < \Omega_1 < \cdots < \Omega_{2m} < \pi$) of $P(e^{j\omega})$ in the stopband.

Until

Satisfy the following condition for a prescribed small constant ϵ (e.g., $\epsilon = 10^{-8}$);

$$\sum_{i=1}^{2m} |\omega_i - \Omega_i| < \epsilon$$

7) Factorize $R(z), B(z)$ to get $Q(z), C(z)$, and use Eq.(25) to construct $H_1(z)$ and $H_2(z)$.

End.

Example 2: We consider a Hilbert transform pair of orthonormal wavelet bases with $M = 9, B = 1$. First, we set $K = 2, L = 2$, so $R = 5$ and the remaining degree of freedom is $2m = 2$. The stopband is set to $\omega_s = 0.57\pi$. We have design the scaling lowpass filters $H_i(z)$ by using the above-mentioned design algorithm. The resulting magnitude response of $H_i(z)$ is shown in solid line in Fig.4. For comparison, the magnitude response of $H_i(z)$ with $K_{max} = 4$ (where $R = 3$) is also shown in dash line in Fig.4. It is seen from Fig.4 that the magnitude response of $H_i(z)$ with $R = 5$ is more sharp than the filter with $R = 3$, at the expense of vanishing moments. The magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ are shown in Fig.5. It is clear that the maximum error obtained by our method is smaller than the conventional method. Moreover, the scaling functions $\phi_i(t)$ and wavelet functions $\psi_i(t)$ are shown in Fig.6, respectively. Finally, the spectrum $\Psi_i(\omega)$ of the wavelet functions and the spectrum $(\Psi_1(\omega) + j\Psi_2(\omega))/2$ are given in Fig.7 and Fig.8. In Fig.8, the spectrum $(\Psi_1(\omega) + j\Psi_2(\omega))/2$ is close to zero in the negative frequency domain ($\omega < 0$).

V. Conclusion

In this paper, we have proposed a new design procedure for DT$\mathcal{C}$WTs by using general IIR filters with numerator and denominator of different degree. That is, we have constructed a class of Hilbert transform pairs of orthonormal wavelet bases based on general IIR filters. First, we have specified the number of zeros at $z = -1$ to get the specified number of vanishing moments, and then used the remaining degree of freedom to improve the frequency selectivity. We have applied Remez exchange algorithm in the stopband to approximate an equiripple magnitude response. Therefore, the optimal solution can be easily obtained by solving the eigenvalue problem through a few iterations. The proposed design procedure is computationally efficient. Furthermore, it is shown that the conventional FIR and IIR orthonormal solutions are only the special cases of our solution proposed in this paper. The advantage of the proposed method is that the number of zeros at $z = -1$ of $H_i(z)$ can be specified arbitrarily.

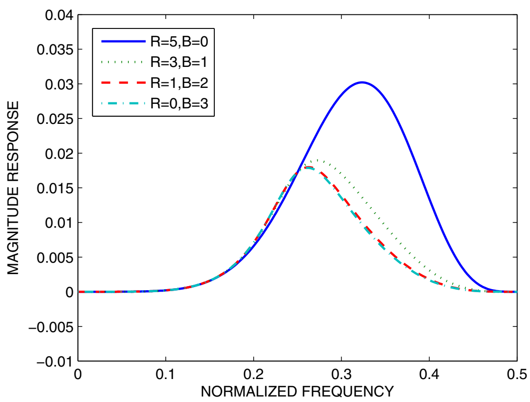

Fig. 3. Magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ in Example 1.

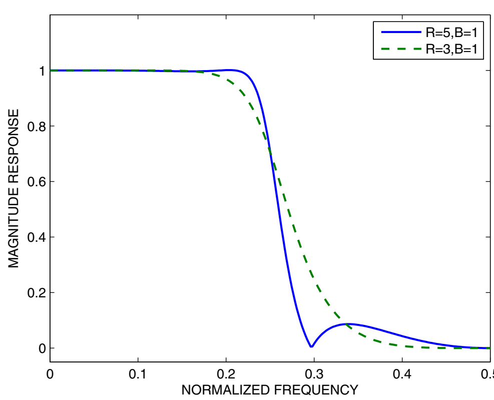

Fig. 4. Magnitude responses of scaling lowpass filters $H_i(z)$ in Example 2.

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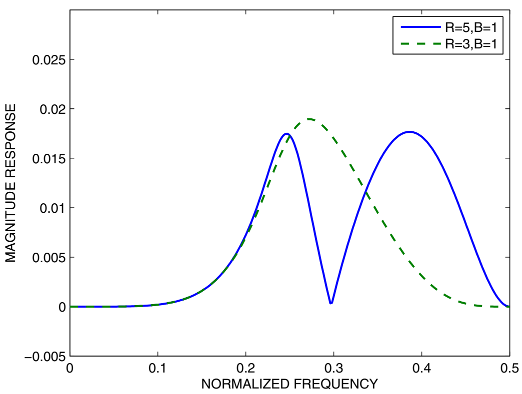

Fig. 5. Magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ in Example 2.

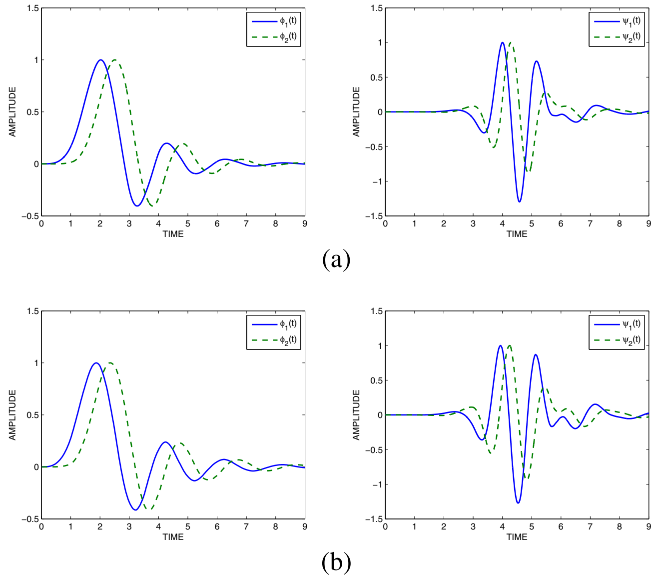

Fig. 6. Scaling functions $\phi_i(t)$ and wavelet functions $\psi_i(t)$ in Example 2. (a) $R = 3, B = 1$, (b) $R = 5, B = 1$.

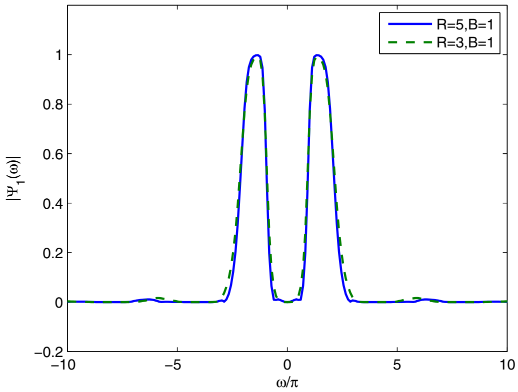

Fig. 7. Magnitude responses of $\Psi_i(\omega)$ in Example 2.

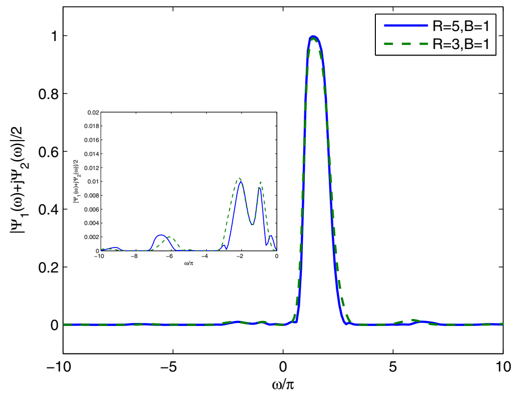

Fig. 8. Magnitude responses of $(\Psi_1(\omega) + j\Psi_2(\omega))/2$ in Example 2.