Design of Full Band IIR Digital Differentiators

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ABSTRACT
This paper presents an efficient method for designing full band IIR digital differentiators in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and then the complex Chebyshev approximation is attained through a few iterations starting from a given initial guess. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm, but also simplifies the interpolation step. One design example is presented and compared with the conventional methods. It can be seen that the design results obtained by using the proposed method are better than that in the conventional methods.

KEY WORDS
IIR Filter, Differentiator, Chebyshev approximation, Remez exchange algorithm, Eigenvalue problem

1 Introduction

Numerical differentiation has been an important signal processing problem, and digital differentiators have been used in a large number of applications [1]∼[11]. The considerable interest in the design of suitable digital differentiators has encouraged the development of various design techniques. The design objective is to get a digital differentiator that meets the specifications in the given sense. Much work has been done, which is mainly devoted to the design of FIR differentiators, since the exactly linear phase response can be easily realized [4],[6],[9],[11]. In contrast, there exists little work regarding IIR differentiators. A design example is given in [7] by using the linear-programming-based method.

In this paper, we propose an efficient method for designing full band IIR digital differentiators in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm [10]. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and then the complex Chebyshev approximation is attained through a few iterations starting from a given initial guess. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm, but also simplifies the interpolation step. Finally, one design example is presented and compared with the conventional methods. It is shown that the design results obtained by using the proposed method are better than that in the conventional methods.

2 IIR Digital Differentiators

The frequency response of an ideal digital differentiator is

$$D(\omega) = j\omega \qquad (|\omega| \leq \pi). \tag{1}$$

In practical design, a constant delay is generally added to obtain a causal solution. Then, the desired frequency response of a digital differentiator is given by

$$H_d(e^{j\omega}) = \omega e^{-j(\tau\omega - \frac{\pi}{2})} \qquad (|\omega| \leq \omega_p), \tag{2}$$

where τ is the given group delay, and ω_p is the cutoff frequency of the interest band. For full band differentiators, $\omega_p = \pi$ and $\tau = K + 0.5$ must be set for digital filters with real coefficients [7], where K is an integer number.

The transfer function $H(z)$ of an IIR digital filter with numerator degree N and denominator degree M is defined by

$$H(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}}, \tag{3}$$

where a_n and b_m are real coefficients, and $b_0 = 1$. The frequency response of $H(z)$ is generally a complex-valued function of the normalized frequency ω:

$$H(e^{j\omega}) = \frac{\sum_{n=0}^{N} a_n e^{-jn\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}}. \tag{4}$$

The complex Chebyshev approximation problem consists in finding the filter coefficients a_n, b_m that will minimize the weighted Chebyshev norm

$$||W(\omega)E(\omega)|| = \max_{|\omega|\le\omega_p} |W(\omega)E(\omega)| \tag{5}$$

of the error function

$$E(\omega) = H(e^{j\omega}) - H_d(e^{j\omega}) \tag{6}$$

among all possible choices of a_n, b_m. To have a constant relative error, we use the weighting function $W(\omega) = 1/|\omega|$ in the interest band [6],[7].

3 Design of IIR Differentiators

In this section, we describe the design of full band IIR digital differentiators based on the eigenvalue problem by using the Remez multiple exchange algorithm. Our aim is to find a set of filter coefficients a_n, b_m in such a way that the error function in Eq.(6) satisfies

$$|E(\omega)| \le \frac{\delta_{max}}{W(\omega)} = \delta_{max}\omega \qquad (0 \le \omega \le \pi), \tag{7}$$

where δ_{max} (> 0) is the maximum error to be minimized. Note that the weighting function $W(\omega)$ becomes ∞ when $\omega = 0$. This means from Eq.(7) that $E(\omega)$ must be zero at $\omega = 0$, that is, $E(0) = 0$.

3.1 Initial Choice

Since the aim is to minimize the maximum error δ_{max}, we pick L frequency points $\bar{\omega}_i$ as shown in Fig.1 and then assume $E(\omega)$ to be zero at these frequency points:

$$E(\bar{\omega}_i) = H(e^{j\bar{\omega}_i}) - H_d(e^{j\bar{\omega}_i}) = 0. \tag{8}$$

When $N + M + 1$ is odd, then $L = (N + M)/2 + 1$, and we pick these frequencies $\bar{\omega}_i$ equally spaced in $[0, \pi)$ from $\bar{\omega}_1 = 0$, as shown in Fig.1(a). Note that $\bar{\omega}_L < \pi$. When $N + M + 1$ is even, $L = (N + M + 1)/2 + 1$, and then we pick $\bar{\omega}_i$ equally spaced in $[0, \pi]$ from $\bar{\omega}_1 = 0$ to $\bar{\omega}_L = \pi$, as shown in Fig.1(b). Since $b_0 = 1$, we substitute Eq.(4) into Eq.(8) and get

$$\sum_{n=0}^{N} a_n e^{-jn\bar{\omega}_i} - j\bar{\omega}_i \sum_{m=1}^{M} b_m e^{-j(m+\tau)\bar{\omega}_i} = j\bar{\omega}_i e^{-j\tau\bar{\omega}_i}. \tag{9}$$

By dividing Eq.(9) into the real and imaginary parts, we have

$$\sum_{n=0}^{N} a_n \cos n\bar{\omega}_i - \bar{\omega}_i \sum_{m=1}^{M} b_m \sin(m+\tau)\bar{\omega}_i = \bar{\omega}_i \sin\tau\bar{\omega}_i, \tag{10}$$

where $i = 1, 2, \dots, L$, and

$$\sum_{n=0}^{N} a_n \sin n\bar{\omega}_i + \bar{\omega}_i \sum_{m=1}^{M} b_m \cos(m+\tau)\bar{\omega}_i = -\bar{\omega}_i \cos\tau\bar{\omega}_i, \tag{11}$$

where $i = 2, 3, \dots, L$ if $N + M + 1$ is odd, and $i = 2, 3, \dots, L - 1$ if $N + M + 1$ is even, since $\bar{\omega}_L = \pi$ and $\tau = K + 0.5$. It is clear that there are a total of $N + M + 1$ equations in Eqs.(10) and (11) whether $N + M + 1$ is odd or even, and hence, we can get an initial solution by solving the linear equations of Eqs.(10) and (11).

3.2 Formulation

By using the obtained initial filter coefficients, we can compute the error function $E(\omega)$ and see that the obtained magnitude response of the weighted error function may not be equiripple. In the following, we will apply the Remez multiple exchange algorithm to obtain an equiripple response. First, we search for all extremal frequencies ω_i in $[0, \pi]$ as follows;

$$0 = \omega_1 < \omega_2 < \cdots < \omega_{L_1} \le \pi, \tag{12}$$

where $L_1 = L + 1$ and $\omega_{L_1} = \pi$ if $N + M + 1$ is odd, and $L_1 = L$ and $\omega_{L_1} < \pi$ if $N + M + 1$ is even, as shown in Fig.1. Note that although $\omega = 0$ is not the extremal frequency, we have set $\omega_1 = 0$, because $W(0) = \infty$ forces $E(0) = 0$. We then compute the phase $\theta(\omega_i)$ of the error function $E(\omega)$ at ω_i, and formulate the condition for $E(\omega)$ as follows;

$$E(\omega_i) = H(e^{j\omega_i}) - H_d(e^{j\omega_i}) = \delta\omega_i e^{j\theta(\omega_i)}, \tag{13}$$

where $\delta(> 0)$ is a magnitude error to be minimized. Substituting Eq.(4) into Eq.(13), we divide Eq.(13) into the real and imaginary parts as

$$\begin{aligned}\sum_{n=0}^{N} a_n \cos n\omega_i - \omega_i \sum_{m=0}^{M} b_m \sin(m+\tau)\omega_i \\ = \delta\omega_i \sum_{m=0}^{M} b_m \cos(m\omega_i - \theta(\omega_i)),\end{aligned} \tag{14}$$

where $i = 1, 2, \dots, L_1$, and

$$\begin{aligned}\sum_{n=0}^{N} a_n \sin n\omega_i + \omega_i \sum_{m=0}^{M} b_m \cos(m+\tau)\omega_i \\ = \delta\omega_i \sum_{m=0}^{M} b_m \sin(m\omega_i - \theta(\omega_i)),\end{aligned} \tag{15}$$

where $i = 2, 3, \dots, L$, since $\omega_{L_1} = \pi$ when $N + M + 1$ is odd. Therefore, there are a total of $N + M + 2$ equations in Eqs.(14) and (15) whether $N + M + 1$ is odd or even. We rewrite Eqs.(14) and (15) in matrix form as

$$\boldsymbol{P}\boldsymbol{a} = \delta\,\boldsymbol{Q}\boldsymbol{a}, \tag{16}$$

where $\boldsymbol{a} = [a_0, a_1, \cdots, a_N, b_0, b_1, \cdots, b_M]^T$, and the elements of the matrices $\boldsymbol{P}, \boldsymbol{Q}$ are given by

$$P_{mn} = \begin{cases} \cos(n-1)\omega_m & (n = 1, 2, \cdots, N+1) \\ -\omega_m \sin(n-N-2+\tau)\omega_m & \\ & (n = N+2, \cdots, N+M+2) \end{cases}, \tag{17}$$

$$Q_{mn} = \begin{cases} 0 & (n = 1, 2, \cdots, N+1) \\ \omega_m \cos((n-N-2)\omega_m - \theta(\omega_m)) & \\ & (n = N+2, \cdots, N+M+2) \end{cases} , \tag{18}$$

if $m = 1, 2, \ldots, L_1$, and

$$P_{mn} = \begin{cases} \sin(n-1)\omega_{m-L_1+1} & (n = 1, 2, \cdots, N+1) \\ \omega_{m-L_1+1} \cos(n-N-2+\tau)\omega_{m-L_1+1} & \\ & (n = N+2, \cdots, N+M+2) \end{cases} , \tag{19}$$

$$Q_{mn} = \begin{cases} 0 & (n = 1, 2, \cdots, N+1) \\ \omega_{m-L_1+1} \sin((n-N-2)\omega_{m-L_1+1} - \theta(\omega_{m-L_1+1})) & \\ & (n = N+2, \cdots, N+M+2) \end{cases} \tag{20}$$

if $m = L_1 + 1, \ldots, N + M + 2$. Therefore, it should be noted that Eq.(16) corresponds to a generalized eigenvalue problem, i.e., δ is an eigenvalue and $\boldsymbol{a}$ is a corresponding eigenvector. In order to minimize δ, we must find the absolute minimum eigenvalue by solving the above eigenvalue problem [10], so that the corresponding eigenvector gives a set of filter coefficients a_n, b_m. Since we are interested in only one eigenvector corresponding to the absolute minimum eigenvalue, this computation can be done efficiently by using the iterative power method without invoking general methods such as the QR technique. By using the obtained filter coefficients, we compute the error function $E(\omega)$ and search for all extremal frequencies ω_i in $[0, \pi]$. As a result, it could be found that the obtained magnitude response may not be equiripple. We then choose L_1 extremal frequencies ω_i as shown in Eq.(12), and calculate the phase $\theta(\omega_i)$ of $E(\omega)$ at ω_i. Therefore, the eigenvalue problem of Eq.(16) can be again solved to obtain a new set of filter coefficients a_n, b_m. The above procedure is iterated until the equiripple response is attained. The design algorithm is shown in detail as follows.

3.3 Design Algorithm

Procedure {Design Algorithm of IIR Digital Differentiators}
Begin

1. Read N, M, and τ.
2. Select L frequency points $\bar{\omega}_i$ as shown in Fig.1.
3. Solve Eqs.(10) and (11) to get an initial solution.
4. Compute $E(\omega)$ to search for all extremal frequencies Ω_i as shown in Eq.(12) and get $\theta(\Omega_i)$.

Repeat

5. Set $\omega_i = \Omega_i$ for $i = 1, 2, \cdots, L_1$.
6. Compute $\boldsymbol{P}$ and $\boldsymbol{Q}$ by using Eqs.(17), (18), (19) and (20), then find the absolute minimum eigenvalue of Eq.(16) to obtain a set of filter coefficients a_n, b_m.
7. Compute $E(\omega)$ to search for all extremal frequencies Ω_i as shown in Eq.(12) and get $\theta(\Omega_i)$.

Until Satisfy the following condition for a prescribed small constant ϵ (in general, $\epsilon = 10^{-6}$):

$$\sum_{i=1}^{L_1} |\Omega_i - \omega_i| \leq \epsilon$$

End.

4 Design Example

In this section, we present one numerical example to demonstrate the effectiveness of the proposed method, and compare the filter performance with the existing design methods. The filter specification is $N = M = 5$, and $\tau = 3.5$, which is the same as *Example 3* in [7]. The initial frequency points $\bar{\omega}_i$ is selected as shown in Fig.1(a). We then obtained a first solution and chose a set of initial extremal frequencies ω_i as shown in Fig.1(a). Starting with these initial extremal frequencies, we obtained an equiripple solution after six iterations. The magnitude response of $E(\omega)$ is shown in Fig.2, and the maximum error is $\delta_{max} = 0.02486$ whereas $\delta_{max} = 0.02592$ in [7]. The magnitude response, phase error and group delay of $H(z)$ are shown in Fig.4, Fig.3 and Fig.5, respectively. The results in [7] are also shown in dotted line for comparison. It can be seen that the proposed method has a smaller group delay error. The pole-zero location of the obtained IIR differentiator is shown in Fig.6 and it is clear that the filter is causal and stable. It has been proved in [3] that to guarantee the causality and stability, a larger group delay should be specified. We have found for this IIR differentiator that when the group delay is set to be larger than $\tau = 2.5$, then the filter becomes causal and stable.

5 Conclusions

In this paper, we have proposed an efficient method for designing full band IIR digital differentiators in the complex Chebyshev sense. The proposed method is based on the formulation of a generalized eigenvalue problem by using the Remez multiple exchange algorithm. Therefore, a set of filter coefficients can be easily obtained by solving the eigenvalue problem to find the absolute minimum eigenvalue, and then the complex Chebyshev approximation is attained through a few iterations starting from a given initial guess. The proposed algorithm is computationally efficient because it not only retains the speed inherent in the Remez exchange algorithm, but also simplifies the interpolation step. Finally, it has been shown through design examples that the design results obtained by using the proposed method are better than that in the conventional methods.

References

[1] S.K.Mitra and J.F.Kaiser, “Handbook for Digital Signal Processing”, John Wiley & Sons, New York, 1993.

[2] B.A.Shenoi, “Magnitude and Delay Approximation of 1-D and 2-D Digital Filters”, Springer, Berlin, 1999.

[3] J.P.Thiran, “Recursive digital filters with maximally flat group delay”, IEEE Trans. Circuit Theory, vol.CT–18, no.6, pp.659–664, Nov. 1971.

[4] T.W.Parks and J.H.McClellan, “Chebyshev approximation for nonrecursive digital filters with linear phase”, IEEE Trans. Circuit Theory, vol.CT–19, no.3, pp.189–194, March 1972.

[5] A.G.Deczky, “Equiripple and minimax (Chebyshev) approximations for recursive digital filters”, IEEE Trans. Acoust., Speech & Signal Processing, vol.ASSP–22, no.2, pp.98–111, April 1974.

[6] S.C.Pei and J.J.Shyu, “Design of FIR Hilbert transformers and differentiators by eigenfilter”, IEEE Trans. Circuits & Systems, vol.CAS–35, no.11, pp.1457–1461, Nov. 1987.

[7] X.Chen and T.W.Parks, “Design of IIR filters in the complex domain”, IEEE Trans. Acoust., Speech & Signal Processing, vol.ASSP–38, no.6, pp.910–920, June 1990.

[8] X.Zhang and H.Iwakura, “Design of IIR digital filters based on eigenvalue problem”, IEEE Trans. Signal Processing, vol.44, no.6, pp.1325–1333, June 1996.

[9] I.W.Selesnick and C.S.Burrus, “Exchange algorithms for linear phase FIR filters and differentiators having flat monotonic passbands and equiripple stopbands”, IEEE Trans. Circuits and Systems II, vol.43, no.9, pp.671–675, Sep. 1996.

[10] X.Zhang, K.Suzuki and T.Yoshikawa, “Complex Chebyshev approximation for IIR digital filters based on eigenvalue problem”, IEEE Trans. Circuits and Systems II, vol.47, no.12, pp.1429–1436, Dec. 2000.

[11] S.C.Pei and P.H.Wang, “Closed-form design of maximally flat FIR Hilbert transformers, differentiators, and fractional delayers by power series expansion”, IEEE Trans. Circuits and Systems I, vol.48, no.4, pp.389–398, April 2001.

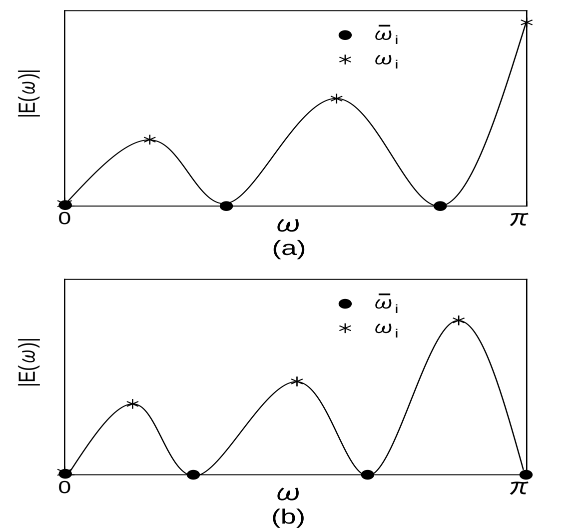

Figure 1. Selection of initial frequency points. (a) $N + M + 1$ is odd, (b) $N + M + 1$ is even.

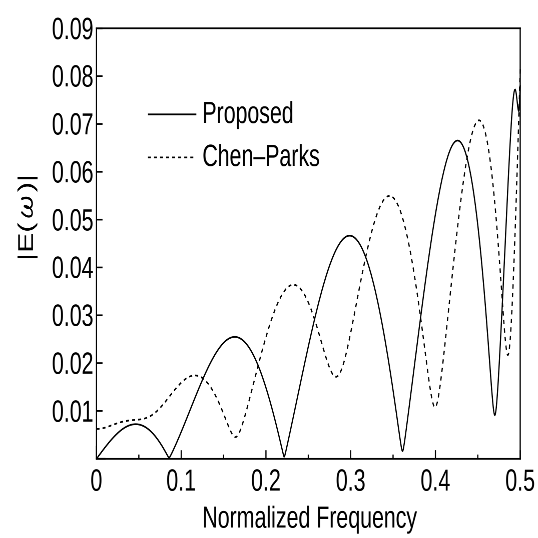

Figure 2. Magnitude responses of $E(\omega)$.

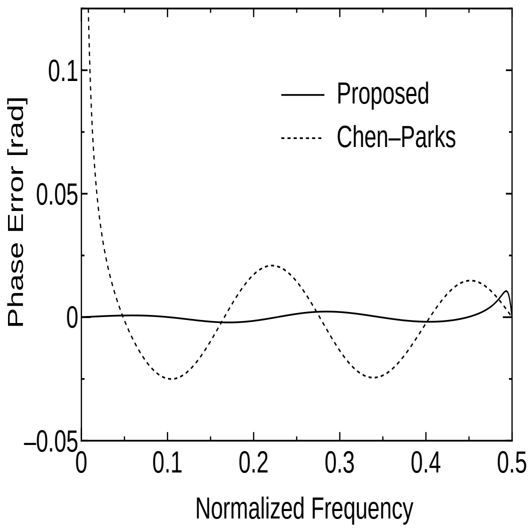

Figure 3. Phase error responses of IIR differentiators.

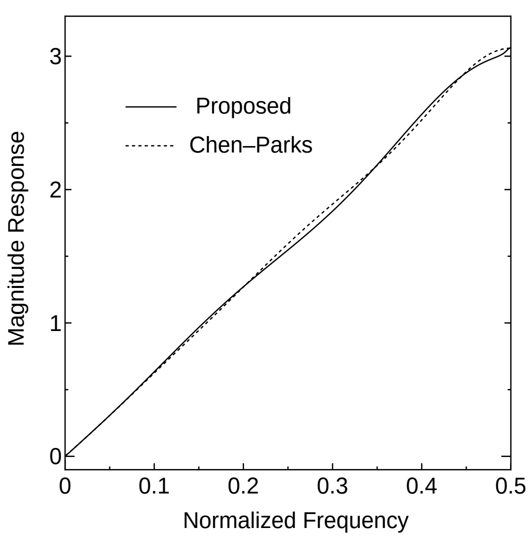

Figure 4. Magnitude responses of IIR differentiators.

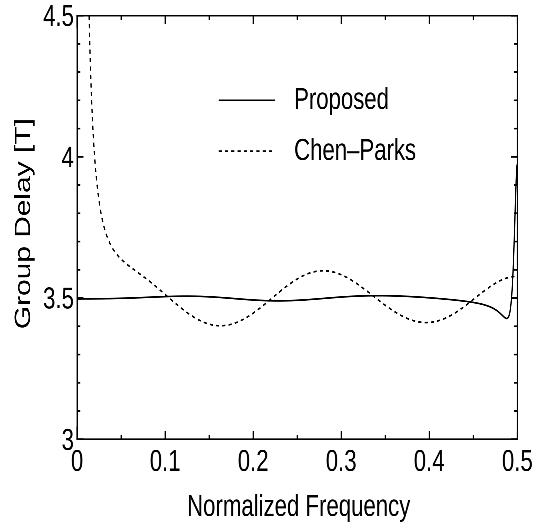

Figure 5. Group delays of IIR differentiators.

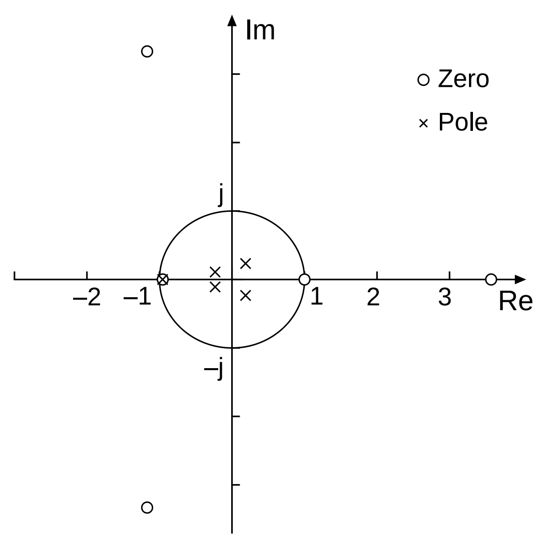

Figure 6. Pole–zero location of IIR differentiator.