CLOSED-FORM DESIGN OF MAXFLAT FRACTIONAL DELAY IIR FILTERS

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ABSTRACT
Fractional delay (FD) filters are an important class of digital filters and are useful in various signal processing applications. In this paper, the design problem of the maxflat FD IIR filters is discussed, and a new closed-form expression for its filter coefficients is presented. The filter coefficients are directly derived by solving a linear system of Vandermonde equations, which are obtained from the flatness conditions of FD filters. The existing maxflat FD FIR and allpass IIR filters are two special cases of the proposed maxflat FD IIR filters. Finally, some examples are designed to demonstrate the effectiveness of the proposed maxflat FD IIR filters.

KEY WORDS
IIR digital filter, fractional delay filter, maxflat frequency response, closed-form design.

1 Introduction

Fractional delay (FD) filters are an important class of digital filters, and have been found numerous applications in signal processing, image processing, and so on [1]. FD filters are required to have the specified fractional delay and flat magnitude response. Conventionally, FIR filters have been used in the design of FD filters [1], [3], [4], [5]. The least-square, minimax (Chebyshev), and maxflat (maximally flat) criterions are used to optimize the frequency response of FD filters. Among these criterions, the maxflat approximation enable us to obtain the closed-form solution of FD FIR filters [3], [4], [5]. Compared with FIR filters, allpass filters, whose magnitude response is constant at all frequencies, can be realized by using IIR filters. Thus only the group delay needs to be optimized for allpass IIR filters. The closed-form solution for the maxflat FD allpass filters has been also given in [2]. However, the design of a general class of maxflat FD IIR filters is still open.

In this paper, we will discuss the design problem of maxflat IIR filters with an arbitrarily specified fractional delay, and give a new closed-form expression for its filter coefficients. We derive a linear system of Vandermonde equations from the flatness conditions of FD filters at $\omega = 0$, and then obtain a set of filter coefficients by directly solving the linear system of Vandermonde equations. The proposed maxflat FD IIR filters include the existing maxflat FD FIR and allpass filters as special cases. Moreover, the proposed maxflat FD IIR filters become causal stable if we choose the desired group delay to be larger than a specific value, which is dependent on the design specification. Finally, some design examples are shown to demonstrate the effectiveness of the proposed maxflat FD IIR filters.

2 FD IIR Filters

Let $H(z)$ be the transfer function of IIR digital filters;

$$H(z) = \frac{\sum_{n=0}^{N} a_n z^{-n}}{\sum_{m=0}^{M} b_m z^{-m}}, \tag{1}$$

where N and M are degrees of numerator and denominator, respectively, and a_n, b_m are real coefficients, where $b_0 = 1$.

The desired frequency response of FD filters is given by

$$H_d(e^{j\omega}) = e^{-j(K+p)\omega}, \tag{2}$$

where K is an integer delay, and p is a fractional delay in the range $[-0.5, 0.5]$.

Let $E(\omega)$ be the weighted error function between $H(e^{j\omega})$ and $H_d(e^{j\omega})$;

$$E(\omega) = W(\omega)[H(e^{j\omega}) - H_d(e^{j\omega})], \tag{3}$$

where $W(\omega)$ is a real and positive weighting function. Therefore, the design problem of FD filters is the approximation of $H(e^{j\omega})$ to $H_d(e^{j\omega})$, that is, the minimization of the error function $E(\omega)$ in the specified criterion, e.g., in the least-square, or minimax, or maxflat sense.

3 Maxflat FD IIR Filters

In this section, we describe the design of FD IIR filters in the maxflat sense. Both the magnitude and group delay responses are required to be flat at $\omega = 0$. Thus, the flatness conditions are given by

$$\begin{cases} |H(e^{j\omega})|\big|_{\omega=0} & = & 1 \\ \dfrac{\partial^r |H(e^{j\omega})|}{\partial \omega^r}\bigg|_{\omega=0} & = & 0 \quad (r=1,2,\cdots,R_1-1) \end{cases}, \tag{4}$$

$$\begin{cases} \tau(\omega)\big|_{\omega=0} & = & K+p \\ \dfrac{\partial^r \tau(\omega)}{\partial \omega^r}\bigg|_{\omega=0} & = & 0 \qquad (r=1,2,\cdots,R_2-2) \end{cases}, \tag{5}$$

where R_1 and R_2 are parameters that control the degree of flatness of the magnitude and group delay responses, respectively. In the following, we will discuss only the case of $R = R_1 = R_2$.

Let $\hat{H}(e^{j\omega}) = H(e^{j\omega})e^{j(K+p)\omega}$. From Eq.(1), we have

$$\hat{H}(e^{j\omega}) = \frac{\sum_{n=0}^{N} a_n e^{j(K+p-n)\omega}}{\sum_{m=0}^{M} b_m e^{-jm\omega}} = \frac{N(\omega)}{D(\omega)}, \tag{6}$$

and the desired frequency response of $\hat{H}(e^{j\omega})$ becomes

$$\hat{H}_d(e^{j\omega}) = 1. \tag{7}$$

It is clear from Eq.(6) that

$$\begin{cases} |\hat{H}(e^{j\omega})| = |H(e^{j\omega})| \\ \hat{\tau}(\omega) = \tau(\omega) - (K+p) \end{cases}, \tag{8}$$

where $|\hat{H}(e^{j\omega})|$ and $\hat{\tau}(\omega)$ are the magnitude and group delay responses of $\hat{H}(e^{j\omega})$, respectively. Therefore, the flatness conditions in Eqs.(4) and (5) are equivalent to

$$\begin{cases} |\hat{H}(e^{j\omega})|\big|_{\omega=0} & = & 1 \\ \dfrac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\bigg|_{\omega=0} & = & 0 \quad (r=1,2,\cdots,R-1) \end{cases}, \tag{9}$$

$$\frac{\partial^r \hat{\tau}(\omega)}{\partial \omega^r}\bigg|_{\omega=0} = 0 \qquad (r=0,1,\cdots,R-2). \tag{10}$$

Note that $R = R_1 = R_2$.

Assume that $\hat{\theta}(\omega)$ is the phase response of $\hat{H}(e^{j\omega})$, then we have

$$\hat{\tau}(\omega) = -\frac{\partial \hat{\theta}(\omega)}{\partial \omega}. \tag{11}$$

Since IIR filters only with real-valued coefficients are considered in this paper, it is known that its phase is 0 at $\omega = 0$, i.e., $\hat{\theta}(0) = 0$. Thus, the condition in Eq.(10) becomes

$$\frac{\partial^r \hat{\theta}(\omega)}{\partial \omega^r}\bigg|_{\omega=0} = 0 \qquad (r=0,1,\cdots,R-1). \tag{12}$$

Theorem 1. *The flatness conditions in Eqs.(9) and (12) are equivalent to*

$$\begin{cases} \hat{H}(e^{j\omega})\big|_{\omega=0} & = & 1 \\ \dfrac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=0} & = & 0 \quad (r=1,2,\cdots,R-1) \end{cases}. \tag{13}$$

Proof. Since $\hat{H}(e^{j\omega}) = |\hat{H}(e^{j\omega})|e^{j\hat{\theta}(\omega)}$, then $\hat{H}(1) = 1$ means $|\hat{H}(1)| = 1$ and $\hat{\theta}(0) = 0$, and vice versa. We have

$$\begin{aligned} \frac{\partial \hat{H}(e^{j\omega})}{\partial \omega} &= \frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} e^{j\hat{\theta}(\omega)} + |\hat{H}(e^{j\omega})| \frac{\partial e^{j\hat{\theta}(\omega)}}{\partial \omega} \\ &= [\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega} + j|\hat{H}(e^{j\omega})| \frac{\partial \hat{\theta}(\omega)}{\partial \omega}] e^{j\hat{\theta}(\omega)}. \end{aligned} \tag{14}$$

Since $|\hat{H}(1)| = 1$ and $\hat{\theta}(0) = 0$, then

$$\frac{\partial \hat{H}(e^{j\omega})}{\partial \omega}\bigg|_{\omega=0} = \frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega}\bigg|_{\omega=0} + j\, \frac{\partial \hat{\theta}(\omega)}{\partial \omega}\bigg|_{\omega=0}. \tag{15}$$

Thus, $\frac{\partial |\hat{H}(e^{j\omega})|}{\partial \omega}\big|_{\omega=0} = 0$ and $\frac{\partial \hat{\theta}(\omega)}{\partial \omega}\big|_{\omega=0} = 0$ are equivalent to $\frac{\partial \hat{H}(e^{j\omega})}{\partial \omega}\big|_{\omega=0} = 0$. Similarly, we have

$$\frac{\partial^2 \hat{H}(e^{j\omega})}{\partial \omega^2}\bigg|_{\omega=0} = \frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2}\bigg|_{\omega=0} + j\, \frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\bigg|_{\omega=0}, \tag{16}$$

which means that $\frac{\partial^2 |\hat{H}(e^{j\omega})|}{\partial \omega^2}\big|_{\omega=0} = 0$ and $\frac{\partial^2 \hat{\theta}(\omega)}{\partial \omega^2}\big|_{\omega=0} = 0$ are equivalent to $\frac{\partial^2 \hat{H}(e^{j\omega})}{\partial \omega^2}\big|_{\omega=0} = 0$.

For $r = 3, 4, \cdots, R-1$,

$$\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\bigg|_{\omega=0} = \frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\bigg|_{\omega=0} + j\, \frac{\partial^r \hat{\theta}(\omega)}{\partial \omega^r}\bigg|_{\omega=0}. \tag{17}$$

Therefore, it can be easily seen that $\frac{\partial^r |\hat{H}(e^{j\omega})|}{\partial \omega^r}\big|_{\omega=0} = 0$ and $\frac{\partial^r \hat{\theta}(\omega)}{\partial \omega^r}\big|_{\omega=0} = 0$ are equivalent to $\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\big|_{\omega=0} = 0$. □

Theorem 2. *The condition in Eq.(13) is equivalent to*

$$\frac{\partial^r N(\omega)}{\partial \omega^r}\bigg|_{\omega=0} = \frac{\partial^r D(\omega)}{\partial \omega^r}\bigg|_{\omega=0} \quad (r=0,1,\cdots,R-1). \tag{18}$$

Proof. From Eq.(6), we have $N(\omega) = \hat{H}(e^{j\omega})D(\omega)$, then $\hat{H}(1) = 1$ means $N(0) = D(0)$. We have

$$\frac{\partial N(\omega)}{\partial \omega} = \frac{\partial \hat{H}(e^{j\omega})}{\partial \omega} D(\omega) + \hat{H}(e^{j\omega}) \frac{\partial D(\omega)}{\partial \omega}. \tag{19}$$

Thus, $\frac{\partial \hat{H}(e^{j\omega})}{\partial \omega}\big|_{\omega=0} = 0$ is equivalent to

$$\frac{\partial N(\omega)}{\partial \omega}\bigg|_{\omega=0} = \frac{\partial D(\omega)}{\partial \omega}\bigg|_{\omega=0}. \tag{20}$$

For $r = 2, 3, \cdots, R-1$, we have

$$\frac{\partial^r N(\omega)}{\partial \omega^r} = \sum_{i=0}^{r} \binom{r}{i} \frac{\partial^{r-i} \hat{H}(e^{j\omega})}{\partial \omega^{r-i}} \frac{\partial^i D(\omega)}{\partial \omega^i}. \tag{21}$$

Since $\hat{H}(1) = 1$ and $\frac{\partial^i \hat{H}(e^{j\omega})}{\partial \omega^i}\Big|_{\omega=0} = 0$ for $i = 1, \cdots, r-1$, then $\frac{\partial^r \hat{H}(e^{j\omega})}{\partial \omega^r}\Big|_{\omega=0} = 0$ is equivalent to

$$\left.\frac{\partial^r N(\omega)}{\partial \omega^r}\right|_{\omega=0} = \left.\frac{\partial^r D(\omega)}{\partial \omega^r}\right|_{\omega=0}. \tag{22}$$

□

According to **Theorem 1 and 2**, the flatness conditions of FD filters have been reduced to Eq.(18). By substituting $N(\omega)$ and $D(\omega)$ in Eq.(6) into Eq.(18), we derive a system of linear equations as follows;

$$\sum_{n=0}^{N} (K+p-n)^r a_n = \sum_{m=0}^{M} (-m)^r b_m, \tag{23}$$

for $r = 0, 1, \cdots, R-1$. By using $b_0 = 1$, Eq.(23) can be rewritten in matrix form as

$$\boldsymbol{V}\boldsymbol{a} = \boldsymbol{u}, \tag{24}$$

where $\boldsymbol{a} = [a_0, a_1, \cdots, a_N, b_1, \cdots, b_M]^T$, $\boldsymbol{u} = [1, 0, \cdots, 0]^T$,

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ K+p & K+p-1 & \cdots & K+p-N & -(-1) & \cdots & -(-M) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (K+p)^{R-1} & (K+p-1)^{R-1} & \cdots & (K+p-N)^{R-1} & -(-1)^{R-1} & \cdots & -(-M)^{R-1} \end{bmatrix}.$$

It should be noted that $\boldsymbol{V}$ is the Vandermonde matrix with distinct elements if $p \neq 0$ [1]. Therefore, there always exists a unique solution if $R = N+M+1$. By using the Cramer's rule, we can obtain the presentation of the filter coefficient as a quotient of two Vandermonde's determinants. Therefore, a closed-form solution is obtained as

$$\begin{cases} a_n = (-1)^{n+1} \dfrac{M!}{n!(N-n)!} \dfrac{\prod_{i=0}^{N}(i-K-p)}{\prod_{i=0}^{M}(i-n+K+p)} \\ b_m = (-1)^m \dfrac{M!}{m!(M-m)!} \prod_{i=0}^{N} \dfrac{i-K-p}{i-m-K-p} \end{cases}. \tag{25}$$

Once the design specification N, M, K and p are given, a set of filter coefficients a_n and b_m can be easily calculated by using Eq.(25) for the maxflat FD IIR filters. If we set $M = 0$ in Eq.(25), then the obtained filters become the maxflat FD FIR filters, which is the same as that proposed in [3], [4] and [5]. Also if $N = M$, we have $a_n = b_{N-n}$ from Eq.(25), thus the resulting filters become the maxflat allpass filters proposed in [2]. Therefore, it is clear that the existing maxflat FD FIR and allpass filters are two special cases of the proposed maxflat FD IIR filters. Therefore, the proposed maxflat FD IIR filters are more general than the conventional maxflat FD (FIR and allpass IIR) filters.

We have found that the proposed maxflat FD IIR filters seldom have its poles at the unit circle. However, the poles may be located outside the unit circle depending on the group delay $K+p$. IIR filters with some poles located outside the unit circle are not causal, but can be divided into the causal and anticausal stable parts that have the poles inside and outside the unit circle respectively, thus, it can be realized in some applications such as image processing and offline processing. When causal stable IIR filters are needed, we have to choose the group delay $K+p$ carefully. It is known in [2] and [6] that allpass filters with $N = M$ become causal stable if the group delay satisfies $K+p > N-1$. Moreover, IIR half-band filters, a special case of FD IIR filters with $p = 0.5$, have been also discussed in [6]. It has been pointed out in [6] that the filters are causal stable when the group delay is larger than a specific value, which is dependent on N and M. Therefore, We have found that the proposed maxflat FD IIR filters become causal stable if we choose the desired group delay $K+p$ to be larger than a specific value, which is dependent on the design specification N and M.

4 Design Examples

In this section, we present several design examples to demonstrate the effectiveness of the proposed maxflat FD IIR filters, and compare the filter performance with the existing maxflat FD FIR and allpass filters.

Example 1: We consider the design of the maxflat FD IIR filters with $N = 8$ and $M = 4$. The degree of flatness is set to $R = N+M+1 = 13$ at $\omega_p = 0$, and the integer delay is $K = 7$. The fractional delay is chosen from $p = -0.5$ to $p = 0.5$ at intervals of $\Delta p = 0.2$. The filter coefficients are calculated by using Eq.(25), and the resulting magnitude and group delay responses are shown in Fig.1 and Fig.2, respectively. It is seen in Fig.1 that the obtained magnitude responses are flat at $\omega = 0$, and are under the influence of the fractional delay p, particularly in the higher frequency. In Fig.2, the group delay responses are also flat at $\omega = 0$, and can be arbitrarily specified. It should be noted that the obtained maxflat FD IIR filters are causal stable when $K+p > 5.80$.

[1]When $p = 0$, the desired delay is integer K, and can be easily realized by z^{-K}. So this case needs not to be considered in the design.

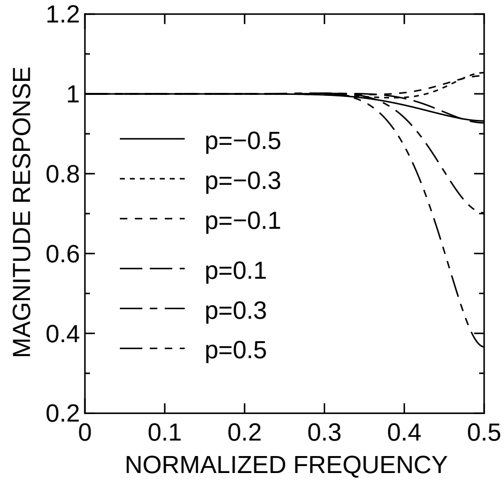

Figure 1. Magnitude responses of the maxflat FD IIR filters in Example 1.

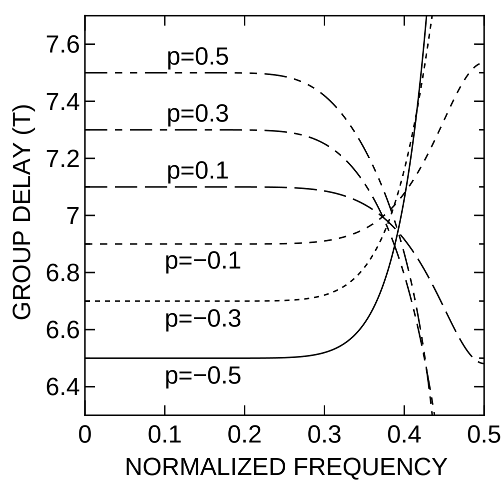

Figure 2. Group delay responses of the maxflat FD IIR filters in Example 1.

Example 2: We consider the design of the maxflat FD IIR filters with $N + M = 10$, and the desired delay $K = 5$ and $p = 0.2$. The degree of flatness is set to $R = N+M+1 = 11$ at $\omega_p = 0$. We have designed the maxflat FD IIR filter with $N = 7$ and $M = 3$ by using Eq.(25). The FD IIR filters with $N = 7$ and $M = 3$ become causal stable when $K + p > 4.64$. The resulting magnitude and group delay responses are shown in the solid line in Fig.3 and Fig.4, respectively. We have also designed the maxflat FD FIR filter with $N = 10$ $(M = 0)$, and allpass filter with $N = M = 5$. Their magnitude and group delay responses are also shown in Fig.3 and Fig.4. It is seen in Fig.3 that the magnitude response of allpass filter is always 1 at all frequencies, whereas the IIR filter with $N = 7$ and $M = 3$ has more flat group delay response than the FD FIR and allpass filters, as shown in Fig.4.

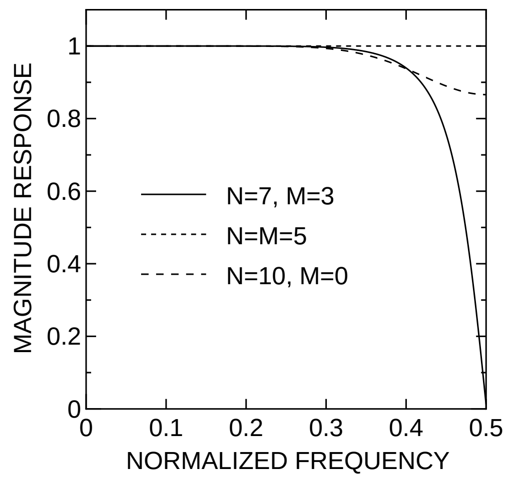

Figure 3. Magnitude responses of the maxflat FD IIR filters in Example 2.

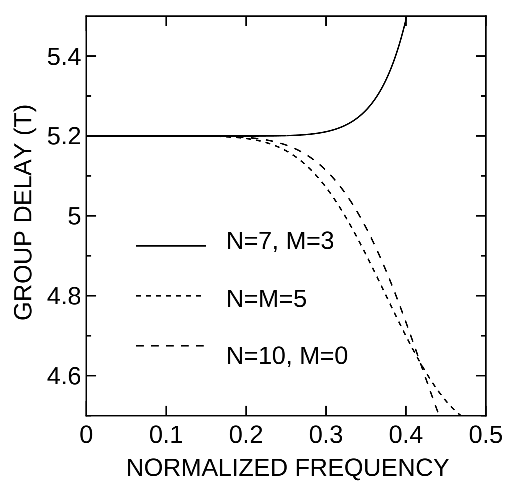

Figure 4. Group delay responses of the maxflat FD IIR filters in Example 2.

5 Conclusion

In this paper, we have discussed the design problem of a general class of FD IIR filters with the maxflat frequency response, and proposed a new closed-form expression for its filter coefficients. The filter coefficients have been directly derived by solving a linear system of Vandermonde equations, which are obtained from the flatness conditions at $\omega = 0$. The proposed maxflat FD IIR filters include the existing maxflat FD FIR and allpass filters as special cases. Moreover, it has found that the maxflat FD IIR filters are causal and stable if the desired group delay is chosen to be larger than a specific value. Finally, some design examples have been shown to demonstrate the effectiveness of the proposed maxflat FD IIR filters.

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