Design of Dual Tree Complex Wavelet Transforms Using IIR Digital Filters

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Abstract: This paper proposes a new design procedure for the dual tree complex wavelet transforms (DT$\mathcal{C}$WTs) using general IIR filters, where two real orthonormal wavelet bases form a Hilbert transform pair. Conventionally, FIR filters had been used to design the DT$\mathcal{C}$WTs, whereas IIR filters were seldom used, although they often require lower computational complexity than FIR filters. In this paper, we use general IIR filters to construct a class of Hilbert transform pairs of orthonormal wavelet bases. The design procedure allows as many zeros as possible to be located at $z = -1$ to obtain the maximum number of vanishing moments. Furthermore, it is shown that the conventional FIR and IIR solutions are only two special cases of our solution proposed in this paper.

Keywords: Dual tree complex wavelet transform, Orthonormal wavelet basis, Hilbert transform pair, IIR filter, Vanishing moment.

1 Introduction

The dual tree complex wavelet transforms (DT$\mathcal{C}$WTs) have been proposed and found to be successful in many applications of signal processing and image processing [4]∼[8], [11]. DT$\mathcal{C}$WTs employ two real wavelet transforms, where one wavelet corresponds to the real part of complex wavelet and the other is the imaginary part. Two wavelet bases are required to form a Hilbert transform pair. Thus, DT$\mathcal{C}$WTs are nearly shift invariant and directionally selective in two and higher dimensions. It has been proven in [7], [9] and [10] that the necessary and sufficient condition for two wavelet bases to form a Hilbert transform pair is the half-sample delay condition between the corresponding scaling lowpass filters. Several design procedures for the Hilbert transform pairs of wavelet bases have been proposed in [4]∼[8] by using FIR filters, which are corresponding to the compactly supported wavelets. In [8], Selesnick had proposed a class of Hilbert transform pairs of wavelet bases, where the corresponding scaling lowpass filters are constructed by using an allpass filter to meet the half-sample delay condition. This design method is simple and effective. The approximation accuracy of the half-sample delay is controlled only by the allpass filter. Thus, the design problem becomes how the scaling lowpass filters to satisfy the orthonormality (or biorthogonality) condition and the regularity of wavelets. In [8], Selesnick had used the maximally flat allpass filter, and then given a class of FIR orthonormal and biorthogonal solutions, and IIR orthonormal solution, where the scaling lowpass filters have as many zeros at $z = -1$ as possible to obtain the maximum number of vanishing moments of wavelets, which results in the maximally flat magnitude response of the scaling lowpass filters. However, for the IIR orthonormal solution proposed in [8], the resulting IIR scaling lowpass filters have the numerator and denominator of the same degree.

In this paper, we propose a new design procedure for DT$\mathcal{C}$WTs using general IIR filters with numerator and denominator of different degree. We restrict ourself to the orthonormal case. That is, we construct a class of Hilbert transform pairs of orthonormal wavelet bases by using general IIR filters. The design procedure allows for scaling lowpass filters as many zeros as possible to be located at $z = -1$ to obtain the maximum number of vanishing moments. Furthermore, it is shown that the FIR and IIR orthonormal solutions proposed in [8] are only two special cases of our solution proposed in this paper. Finally, one design example is presented to demonstrate the effectiveness of our design procedure.

2 Dual Tree Complex Wavelet Transforms

DT$\mathcal{C}$WTs was first introduced by Kingsbury in 1998 [4]∼[6]. DT$\mathcal{C}$WTs employ two real DWTs (discrete wavelet transforms); the first DWT gives the real part of DT$\mathcal{C}$WTs and the second DWT gives the imaginary part. The first wavelet basis is required to be the Hilbert transform of the second wavelet basis.

It is well-known that orthonormal wavelet bases can be generated by two-band orthogonal filter banks $\{H_i(z),\ G_i(z)\}$, where $i = 1, 2$. We assume that $H_i(z)$ is lowpass filter, and $G_i(z)$ is highpass. The orthonormality condition of $H_i(z)$ and $G_i(z)$ is given by

$$\begin{cases} H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 2 \\ G_i(z)G_i(z^{-1}) + G_i(-z)G_i(-z^{-1}) = 2 \\ H_i(z)G_i(z^{-1}) + H_i(-z)G_i(-z^{-1}) = 0 \end{cases}. \tag{1}$$

Let $\phi_i(t), \psi_i(t)$ be the corresponding scaling and wavelet functions, respectively. The dilation and wavelet equations give the scaling and wavelet functions;

$$\begin{cases} \phi_i(t) = \sqrt{2}\sum_n h_i(n)\phi_i(2t-n) \\ \psi_i(t) = \sqrt{2}\sum_n g_i(n)\phi_i(2t-n) \end{cases}, \tag{2}$$

where $h_i(n)$ and $g_i(n)$ are the impulse responses of $H_i(z)$ and $G_i(z)$, respectively.

It has been proven in [7], [9] and [10] that two wavelet functions $\psi_1(t)$ and $\psi_2(t)$ form a Hilbert transform pair;

$$\psi_2(t) = \mathcal{H}\{\psi_1(t)\}, \tag{3}$$

that is

$$\Psi_2(\omega) = \begin{cases} -j\Psi_1(\omega) & (\omega > 0) \\ j\Psi_1(\omega) & (\omega < 0) \end{cases}, \tag{4}$$

if and only if two scaling lowpass filters satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{5}$$

where $\Psi_i(\omega)$ are the Fourier transform of $\psi_i(t)$. This is the so-called half-sample delay condition between two scaling lowpass filters. Equivalently, the scaling lowpass filters should be offset from one another by a half sample. Eq.(5) is the necessary and sufficient condition for two orthonormal wavelet bases to form a Hilbert transform pair.

3 Hilbert Transform Pairs of Orthonormal Wavelet Bases Composed of Allpass Filter

It is known that the transfer function of an allpass filter $A(z)$ is defined by

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)}, \tag{6}$$

where

$$D(z) = 1 + \sum_{n=1}^{L} d(n)z^{-n}, \tag{7}$$

where L is the degree of $A(z)$ and $d(n)$ are real filter coefficients. Therefore, the phase response $\theta(\omega)$ of $A(z)$ is given by

$$\theta(\omega) = -L\omega + 2\tan^{-1}\frac{\sum_{n=1}^{L} d(n)\sin(n\omega)}{1 + \sum_{n=1}^{L} d(n)\cos(n\omega)}. \tag{8}$$

In [8], Selesnick has proposed that the scaling lowpass filters $H_1(z)$ and $H_2(z)$ have the following form;

$$\begin{cases} H_1(z) = F(z)D(z) \\ H_2(z) = F(z)z^{-L}D(z^{-1}) \end{cases}, \tag{9}$$

and $G_i(z) = (-z)^{-M}H_i(-z^{-1})$ for $i = 1, 2$, where M is the degree of $H_i(z)$ and is an odd number.

Since $H_1(z)$ and $H_2(z)$ have the common divisor $F(z)$, we have

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z). \tag{10}$$

Therefore, if $A(z)$ in Eq.(6) is an approximate half-sample delay;

$$A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}} \qquad (-\pi < \omega < \pi), \tag{11}$$

then the half-sample delay condition in Eq.(5) is achieved approximately. Thus, two wavelet bases form an approximate Hilbert transform pair.

There exist many design methods for allpass filters to approximate a fractional delay response, for example, the maximally flat, least squares [2], equiripple approximations [3], and so on. In [8], the maximally flat fractional delay allpass filter was adapted, and $\omega = 0$ was chosen for the point of approximation. However, the approximation error will increase as ω goes away from the point of approximation in the maximally flat approximation. Thus, it will be better if the minimax (Chebyshev) phase approximation of allpass filters is used, e.g., [3].

Once $A(z)$ is determined, $F(z)$ needs to be designed for $H_1(z)$ and $H_2(z)$. To obtain wavelet bases with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1 + z^{-1})^K. \tag{12}$$

Thus

$$\begin{cases} H_1(z) = Q(z)(1 + z^{-1})^K D(z) \\ H_2(z) = Q(z)(1 + z^{-1})^K z^{-L}D(z^{-1}) \end{cases}. \tag{13}$$

It is clear that $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(1 + z)^K(1 + z^{-1})^K D(z)D(z^{-1}). \end{aligned} \tag{14}$$

Let $Q(z)$ be a FIR filter and defining

$$R(z) = Q(z)Q(z^{-1}) = \sum_{n=-R}^{R} r(n)z^{-n}, \tag{15}$$

$$S(z) = (z + 2 + z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \tag{16}$$

where $r(n) = r(-n)$ for $1 \le n \le R$ and $s(n) = s(-n)$ for $1 \le n \le L + K$, we can write the orthonormality condition in Eq.(1) as

$$\sum_{k=I_{min}}^{I_{max}} s(2n - k)r(k) = \begin{cases} 1 & (n = 0) \\ 0 & (1 \le n \le \dfrac{R + L + K}{2}) \end{cases}, \tag{17}$$

where $I_{min} = \max\{-R, 2n - L - K\}$ and $I_{max} = \min\{R, 2n + L + K\}$. Note that $P(z)$ is a halfband filter, thus $R + L + K = M$ is an odd number, where $M = R + L + K$ is the degree of $H_i(z)$. Since $r(n) = r(-n)$, there are $(M + 1)/2$ equations with respect to $(R + 1)$ unknown coefficients $r(n)$ in Eq.(17). Therefore, it is clear that we can obtain the only solution $r(n)$ if $(M + 1)/2 = R + 1$. In [8], Selesnick had chosen $R = L + K - 1$ and obtained the filter of minimal degree for given L and K, which is equivalent to the maximal K ($K_{max} = R - L + 1 = (M + 1)/2 - L$) for given R and L. Thus the scaling lowpass filters have the maximally flat magnitude response, resulting in the maximum number of vanishing moments of wavelets. This is the FIR orthonormal solution proposed in [8].

4 General IIR orthonormal solution

IIR filters can also be used to construct a class of Hilbert transform pairs of orthonormal wavelet bases. In [8], Selesnick has chosen

$$F(z) = \frac{(1+z^{-1})^K}{C(z^2)}, \tag{18}$$

then

$$\begin{cases} H_1(z) = \dfrac{(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \dfrac{(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)} \end{cases}. \tag{19}$$

Therefore, $H_1(z)$ and $H_2(z)$ have the same product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= \frac{(1+z)^K(1+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})}. \end{aligned} \tag{20}$$

Defining

$$B(z) = C(z)C(z^{-1}) = \sum_{n=-B}^{B} b(n)z^{-n}, \tag{21}$$

where $b(n) = b(-n)$ for $1 \leq n \leq B$, from the orthonormality condition in Eq.(1), we have $B = \lfloor \frac{L+K}{2} \rfloor$ and

$$b(n) = s(2n), \tag{22}$$

where $\lfloor x \rfloor$ means the largest integer not greater than x. This is the IIR orthonormal solution proposed in [8]. It is clear that the numerator and denominator of $H_i(z)$ are of degree $M = L + K$ and $2B = 2\lfloor \frac{L+K}{2} \rfloor$ respectively, which are the (almost) same.

In this paper, we use general IIR filters with numerator and denominator of different degree to construct a class of Hilbert transform pairs of orthonormal wavelet bases. We choose

$$F(z) = \frac{Q(z)(1+z^{-1})^K}{C(z^2)}, \tag{23}$$

thus

$$\begin{cases} H_1(z) = \dfrac{Q(z)(1+z^{-1})^K D(z)}{C(z^2)} \\ H_2(z) = \dfrac{Q(z)(1+z^{-1})^K z^{-L} D(z^{-1})}{C(z^2)} \end{cases}. \tag{24}$$

We have the product filter $P(z)$;

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= \frac{Q(z)Q(z^{-1})(1+z)^K(1+z^{-1})^K D(z)D(z^{-1})}{C(z^2)C(z^{-2})}. \end{aligned} \tag{25}$$

From the orthonormality condition in Eq.(1), we have

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} b(n) & (0 \leq n \leq B) \\ 0 & (B < n \leq \dfrac{M}{2}) \end{cases}, \tag{26}$$

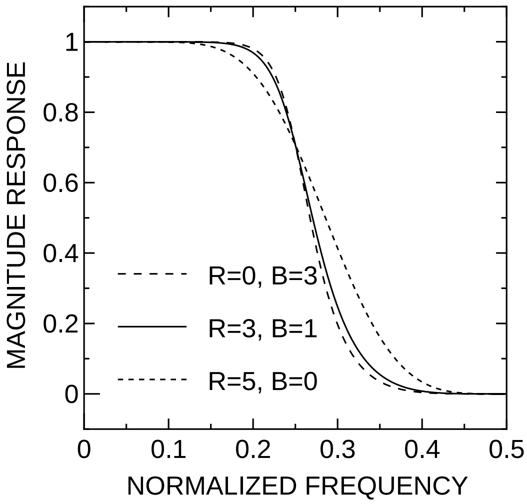

Fig. 1. Magnitude responses of lowpass filters $H_i(z)$.

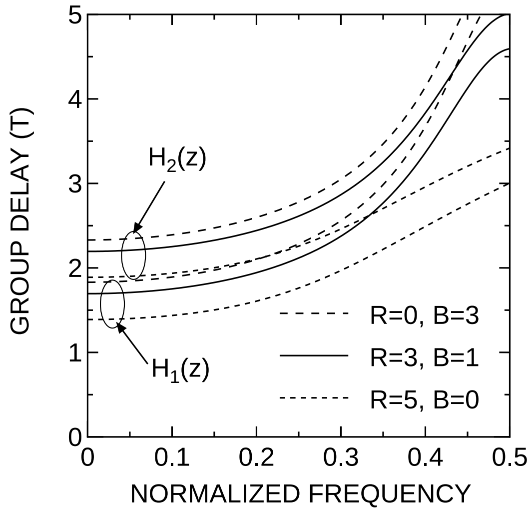

Fig. 2. Group delays of lowpass filters $H_i(z)$.

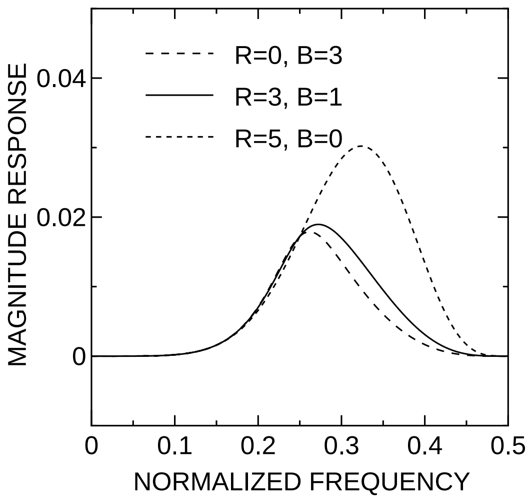

Fig. 3. Magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$.

where $B \leq \frac{M}{2}$, and $M = R + L + K$ is the degree of numerator of $H_i(z)$ and is an odd number if $R \neq 0$. Assuming $b(0) = 1$, there are $\lfloor \frac{M}{2} \rfloor + 1$ equations with respect to $(R + B + 1)$ unknown coefficients $r(n)$ and $b(n)$ in Eq.(26). Therefore, it is clear that the only solution exists if $\lfloor \frac{M}{2} \rfloor + 1 = R + B + 1$. When $R \neq 0$, $R + 2B = L + K - 1$, because M is an odd number. If we choose $B = 0$, then $R = L + K - 1$, which is the FIR orthonormal solution proposed in [8]. If we choose $R = 0$, then $B = \lfloor \frac{M}{2} \rfloor = \lfloor \frac{L+K}{2} \rfloor$, which is the IIR orthonormal solution proposed in [8]. Therefore, the FIR and IIR orthonormal solutions in [8] are only two special cases of our solution using general IIR filters when $B = 0$ and $R = 0$. Similar to the way proposed in [8], we can obtain $Q(z)$ and $C(z)$ from $R(z)$ and $B(z)$ by using a spectral factorization approach. To obtain stable IIR filters, we use the minimum-phase spectral factor so that $C(z)$ has all zeros in the unit circle. Then we have

$$\begin{cases} G_1(z) = \dfrac{z^{-R}Q(-z^{-1})(1 - z^{-1})^K z^{-L} D(-z^{-1})}{C(z^2)} \\ G_2(z) = \dfrac{z^{-R}Q(-z^{-1})(1 - z^{-1})^K D(-z)}{C(z^2)} \end{cases}. \tag{27}$$

5 Design Example

In this section, we present one example to demonstrate the effectiveness of our design procedure proposed in this paper. We consider a Hilbert transform pair of orthonormal wavelet bases with $K = 4$ and $L = 2$ as in [8]. Firstly, we have designed the scaling lowpass filters $H_i(z)$ with $R = 3$ and $B = 1$. The resulting magnitude response of $H_i(z)$ with $M = 9$ is shown in solid line in Fig.1, Next, we designed $H_i(z)$ with $R = 5, B = 0$ (FIR) and $R = 0, B = 3$ (IIR) proposed in [8]. The magnitude responses of $H_i(z)$ are also shown in Fig.1. It is seen in Fig.1 that two IIR filters have more sharp magnitude responses than the FIR filter. Their group delays are given in Fig.2, and it is clear that $H_1(z)$ and $H_2(z)$ satisfy the half-sample delay condition. Moreover, the magnitude responses of $H_1(e^{j\omega})e^{-j\frac{\omega}{2}} - H_2(e^{j\omega})$ are shown in Fig.3. It is seen in Fig.3 that the maximum errors of IIR filters are smaller than that of FIR filter. Finally, the obtained scaling and wavelet functions $\phi_i(t), \psi_i(t)$ are given in Fig.4, and the spectrum $\Psi_i(\omega)$ of the wavelet functions $\psi_i(t)$ are shown in Fig.5. The spectrum $\Psi_1(\omega) + j\Psi_2(\omega)$ of $\psi_1(t) + j\psi_2(t)$ are given in Fig.6, which is more close to zero in the negative frequency domain ($\omega < 0$) for the IIR cases.

6 Conclusion

In this paper, we have proposed a new design procedure for the dual tree complex wavelet transforms using general IIR filters with numerator and denominator of different degree. That is, we have constructed a class of Hilbert transform pairs of orthonormal wavelet bases by using general IIR filters, including the FIR and IIR orthonormal solutions proposed in [8] as two special cases of $B = 0$ and $R = 0$. The design procedure allows for scaling lowpass filters as many zeros as possible to be located at $z = -1$ to obtain the maximum number of vanishing moments of wavelets. Furthermore, one design example has been shown to demonstrate the effectiveness of our design procedure proposed in this paper.

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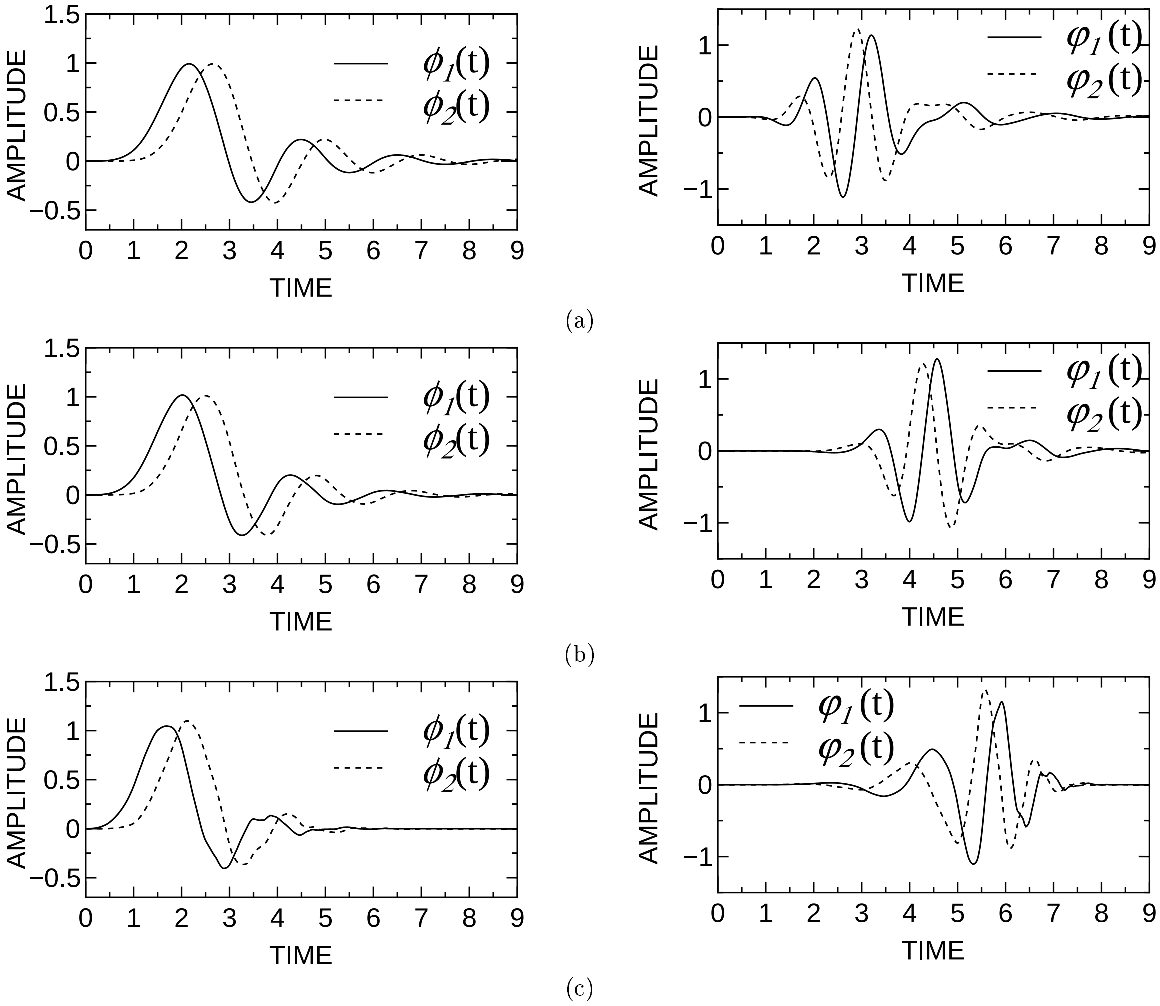

Fig. 4. Scaling and wavelet functions $\phi_i(t), \psi_i(t)$. (a) $R = 0, B = 3$, (b) $R = 3, B = 1$, (c) $R = 5, B = 0$

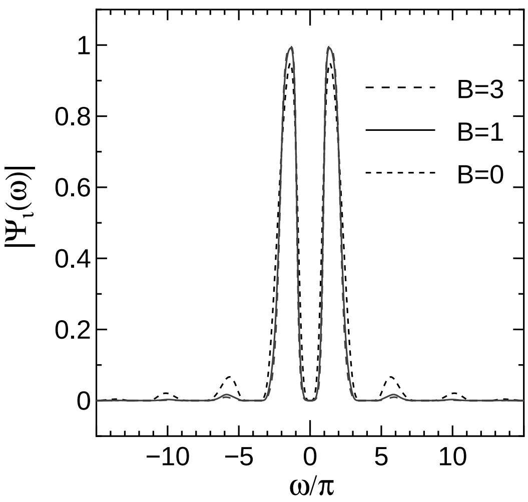

Fig. 5. Magnitude responses of $\Psi_i(\omega)$.

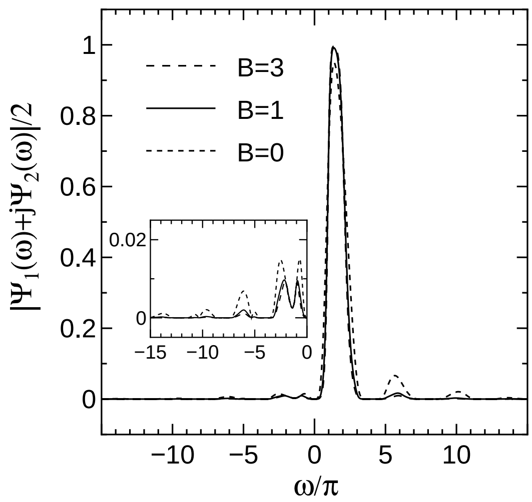

Fig. 6. Magnitude responses of $(\Psi_1(\omega) + j\Psi_2(\omega))/2$.